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Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations

H. BERESTYCKI

Université Paris VI
Laboratoire d'Analyse Numérique (55-65,5e)
4 Pl Jussieu
F-75252 Paris Cedex 05

L. NIRENBERG

Courant Institute
New York Univ.
251 Mercer St.
New York, N.Y. 10012
U.S.A.

Dedicated to I.M. Gelfand on his seventy-fifth birthday

Abstract. *This paper investigates certain properties of solutions of equations of the form*

$$\Delta u + f(x, u, \nabla u) = 0$$

in a bounded domain Ω *which is convex in the* x_1 *direction. Under various conditions on* f *and on a solution* u *it is shown that* u *is increasing in* x_1 *in the «left half» of* Ω*, or in all of* Ω*. Symmetry (in* x_1*) of some solutions is proved. Also antisymmetry results are obtained. The paper may be considered as an extension of* [4].

Key Words: Elliptic partial differential equations. Parabolic equations, maximum principle, monotonicity, symmetry, antisymmetry, uniqueness.

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1. INTRODUCTION

In [4] properties of monotonicity and symmetry were established for positive solutions u, vanishing on the boundary, of elliptic equations, using the maximum principle and the method of moving planes. The method is due to A.D. Alexandroff and was then used by J. Serrin [8].

A typical result in [4] is the following:

THEOREM A. *Let Ω be a bounded domain in R^n with C^2 boundary which is convex in the x_1 direction and symmetric about $x_1 = 0$. Let u be a positive solution of*

$$\Delta u + f(u) = 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

where $f \in C^1$. Then $u_{x_1} = u_1 > 0$ for $x_1 < 0$ and u is symmetric in x_1.

In case Ω is a ball it then follows that u is radially symmetric and $u_r < 0$.

The symmetry is a by product of monotonicity in the x_1 direction–proved by the method of moving planes as follows. For λ real let T_λ be the plane $x_1 = \lambda$, and

$$\Sigma(\lambda) = \{x \in \Omega; x_1 < \lambda\}.$$

Let x^λ denote the reflection of x in T_λ, i.e. x^λ has the same coordinates as x except for $x_1^\lambda = 2\lambda - x_1$. The monotonicity result which is proved in [4] is

$$w(x) = u(x^\lambda) - u(x) > 0 \text{ for } x \in \{\Sigma(\lambda), \lambda < 0\}.$$

In the limit we have $u(x^0) \geqslant u(x)$ for $x \in \Sigma(0)$. Since Ω and the equation are symmetric in x_1, the opposite inequality follows, and hence symmetry.

Symmetry properties of solutions in all of R^n were also proved in [4] and [5].

The authors of this paper extended some of the monotonicity and symmetry results to semilinear equations in infinite cylinders. In the process of doing that they discovered some generalisations of results of [4] in bounded domains. These are the subject of this paper. So this should be regarded as a continuation of [4]. We no longer assume that $u > 0$ and $u = 0$ on $\partial\Omega$. For convenience we confine ourselves to equations of the form

$$\Delta u + f(x, u, \nabla u) = 0. \tag{1.1}$$

It is always assumed that $f(u, x, p)$ is continuous in all variables and Lipschitz continuous in (u, p). The arguments apply to some more general elliptic equations.

In fact after the work was completed Mr. Li, Cong Ming discovered that they could be adapted to handle fully nonlinear second order elliptic equations. In particular he can improve one of the main results, Theorem 2.1′ of [4]: he shows that the result holds even if the hypothesis (b) is dropped.

Recently Caffarelli, Gidas and Spruck [11] have used the method of moving planes, rather a measure theoretic variation, to obtain very strong results on behaviour of solutions, of certain elliptic equations, near isolated singular points.

We will now describe some of our results (in R^n we represent $x = (x_1, y)$, $y = (x_2, \dots, x_n)$. In section 2 and 3 we assume that f satisfies for $x_1 < x_1'$, $x_1 + x_1' < 0$,

$$f(x_1, y, u, p_1, \dots, p_n) \leqslant f(x_1', y, u, -p_1, p_2, \dots, p_n) \text{ if } p_1 \geqslant 0. \tag{1.2}$$

Here is a symmetry result which will be proved in section 2. We consider a finite cylinder

$$\Omega = S_a = \{(x_1, y) \in R^n;\ |x_1| < a, y \in \omega\} \tag{1.3}$$

where ω is a bounded domain in R^{n-1} with smooth boundary.

THEOREM 1.1. *Let u be a C^2 solution of (1.1) in $\overline{\Omega}$ and assume that the boundary values of u are symmetric in x_1 and satisfy*

$$u_1(x_1, y) \geqslant 0 \text{ for } -a < x_1 < 0, \quad y \in \partial\omega. \tag{1.4}$$

Assume also that

$$\begin{cases} u(-a, y) \leqslant u(x_1, y) \text{ for } -a < x_1 < a,\ y \in \omega \\ \text{and } \forall x_1 \text{ in } (-a, a),\ \exists y \in \omega \textit{ such that strict inequality holds.} \end{cases} \tag{1.5}$$

Assume that f satisfies (1.2) and that it is symmetric in (x_1, p_1). Then u is a symmetric function of x_1 and $u_1 > 0$ if $x_1 < 0$.

The theorem will be derived from a monotonicity result, Theorem 2.1, for solutions u of (1.1) in S_a satisfying (1.5). Theorem 2.2 contains a similar result under Neumann boundary conditions on the curved part of ∂S_a.

Here is an example in which the theorem applies. In the cylinder Ω let z be a solution of

$$\Delta z + g(z) = 0$$

$$z = \phi \text{ on } \partial\Omega$$

where $\phi \geqslant 0$, ϕ is symmetric in x_1, $\phi_1 \geqslant 0$ for $x_1 < 0$, $y \in \partial\omega$, and $\phi = 0$ at the ends: $x_1 = \pm a$. We assume g is a C^1 nonnegative function. For $\sigma(y)$ a smooth function of y, the function $u = z - \sigma(y)$, satisfies (1.1) with $f = g(u + \sigma(y)) + \Delta_y \sigma$.

In section 3 we extend the x_1-monotonicity results to general domains Ω with smooth boundary which are convex in the x_1 direction. Again we replace the conditions of [4], that $u > 0$ in Ω, $u = 0$ in $\partial\Omega$, by the condition that on any segment in Ω parallel to the x_1 axis, u is greater than its value at the left end point lying on $\partial\Omega$. In addition, assuming u is less than its value at the right end point in $\partial\Omega$ we prove monotonicity (in x_1) in the full domain Ω.

Here is a special case of Theorem 3.1$'$:

THEOREM 1.1$'$: *Let Ω be a bounded domain, with smooth boundary, which is convex in the x_1 direction and symmetric in x_1. In $\bar{\Omega}$ let u be a C^3 solution of*

$$\Delta u + f(u) = 0$$

$$u = \phi \quad on \quad \partial\Omega,$$

where f is Lipschitz continuous Assume that ϕ is symmetric in x_1 and that for $x_1 \leqslant 0$, wherever the normal ν to $\partial\Omega$ has $\nu_1 = 0$, there $\phi_1 \leqslant 0$. Assume also that if $(x_1, y) \in \Omega$, $(x_1', y) \in \partial\Omega$ then

$$u(x_1, y) > \phi(x_1', y).$$

Then u is symmetric in x_1 and $u_1 > 0$ for $x_1 < 0$.

L. Caffarelli indicated to us a different and very simple argument for proving x_1-monotonicity under the condition (in place of (1.2)): f is nondecreasing in x_1. See Theorem 3.5 and its corollary. In section 4 we modify his argument to prove monotonicity under the weaker condition

$$f(x, u, p) \text{ is nondecreasing in } x_1 \quad \text{for } p_1 \geqslant 0. \tag{1.6}$$

Here is a simplified form of Theorem 4.1 in the cylinder $\Omega = S_a$.

THEOREM 1.2. *Let $u \in C^2(\bar{\Omega})$ be a solution of (1.1) with f satisfying (1.6) and Lipschitz continuous in (x_1, u, p). Assume that $u_1 \geqslant 0$ on $\partial\Omega$ and that*

$$u(-a, y) < u(x_1, y) < u(a, y) \quad \forall y \in \omega, -a < x_1 < a. \tag{1.5$'$}$$

Then $u_1 > 0$ in Ω. Furthermore if $\underline{u}$ is any solution of (1.1) agreeing with u on $\partial\Omega$ and satisfying (1.5)$'$ then $\underline{u} = u$.

We conclude the introduction with some remarks about the maximum principle for solutions in a domain Ω of an elliptic inequality (we use summation convention)

$$a_{ij}(x)\, u_{x_i x_j} + b_i(x)\, u_{x_i} + c(x)\, u \geqslant 0. \tag{1.7}$$

Uniform ellipticity is assumed:

$$a_{ij}\xi_i\xi_j \geqslant c_0 \,|\,\xi\,|^2, \;\; c_0 > 0. \tag{1.8}$$

We will constantly refer to results from [4]: the Maximum Principle and its Corollary as well as Lemma H on pages 212-3 of [4], and also to the extended Hopf boundary lemma at a corner, Lemma S, in [4]. In addition we will use Lemma A.1 there. In all of these results, as stated in [4], it is assumed that $u \in C^2(\bar{\Omega})$ and that the coefficients in (1.7) are continuous (for Lemma S further regularity of the $\{a_{ij}\}$ is required).

For various applications it is important to be able to relax these smoothness requirements. In [1] Amick and Fraenkel have extended (and used) these results for equations in divergence form, with merely bounded measurable coefficients. For equations (1.7) in nondivergence form we point out that it is enough to suppose that $u \in W^{2,p}$, $p > n$, and that the coefficients in (1.7) belong to L^∞. Namely, the Maximum Principle and its Corollary, and Lemma H of [4] hold for such solutions. The proofs of these assertions proceed as in the classical cases, with the aid of the usual barrier functions, but using the Bony maximum principle [3]. Recall its statement (see P.L. Lions [6] for an extension):

Bony Max. Princ.: Let $u \in W^{2,p}$ $p > n$, in a neighbourhood of the origin in R^n, and have a local maximum at the origin. Then

$$\liminf_{x \to 0} \operatorname{ess} \; \Sigma a_{ij}(x) u_{ij}(x) \leqslant 0$$

where $\{a_{ij}\}$ is a nonnegative matrix belonging to $L^\infty_{\text{loc}}(\Omega)$.

Furthermore, Lemma S of [4] holds for $C^2(\bar{\Omega})$ solutions of (1.7) in case $a_{ij} \in C^2(\bar{\Omega})$, and the coefficients b_i and c are in L^∞.

New ideas are involved in proving the new results. A new ingredient in our application of the method of moving planes is the use of parabolic inequalities – and the corresponding maximum principle. This came as a surprise to us. In particular we will need an analogue of the corollary on page 213 of [4]. We state it only in the simplest form. In R^{n+1} with coordinates (x, t), $x \in R^n$, let V be a bounded domain lying in $t < T$. In $\bar{V}$ we consider a function w, with continuous derivatives up to second order in x and first order in t, satisfying a (degenerate) parabolic inequality:

$$(L - \beta\partial_t)w = a_{ij}(x, t)\, w_{x_i x_j} + b_i(x, t) w_{x_i} + c(x, t) w - \beta(x, t) w_t \geqslant 0. \tag{1.9}$$

Here the a_{ij} are continuous and satisfy (1.8) and the other coefficients are all in L^∞; in addition $\beta \geqslant 0$.

PROPOSITION 1.1. *Assume that at every point of*

$$J = \overline{\partial V \cap \{t < T\}}$$

we have $w \leqslant 0$. *If* V *lies in a sufficiently narrow region* $0 < a + x_1 < \epsilon$ *then* $w \leqslant 0$ *in* V.

Here ϵ depends in the constant c_0 in (1.8) and on the L^∞ norms of the coefficients b_i, c, β.

The proposition will be proved in the Appendix together with analogues of Lemmas H and S which are stated in section 4. See also Lemma A.1.

We wish to express our thanks to L. Caffarelli for several useful remarks and to Li, Cong Ming for his remarks and simplifications of some of the proofs.

2. MONOTONICITY

In this section we will present generalizations of one of the basic results, Theorem 2.1 of [4] (1). The reader should note however that, here, the moving planes will move in the direction of increasing x_1 (in [4] it was the other way). So our conditions will look a bit different from those in [4]. As we have remarked, we do not assume $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω. Rather we assume, essentially, that on any open interval in Ω parallel to the x_1 axis u, is greater than its value at the left end point (i.e. smallest x_1) on $\partial\Omega$.

In this section we consider the simplest geometry, the finite cylinder $\Omega = S_a$ of § 1:

$$S_a = \{(x_1, y) \in R^n ; | x_1 | < a, y \in \omega\}. \tag{2.1}$$

In $[-a, a) \times \overline{\omega}$ we consider a C^2 solution of

$$\Delta u + f(x, u, \nabla u) = 0. \tag{2.2}$$

We will prove monotonicity of u in x_1 for $x_1 < 0$ assuming (1.2):

$$\begin{cases} \text{for } x_1 < x_1', \ x_1 + x_1' < 0, \ \text{and } p_1 \geqslant 0, \\ f(x_1, y, u, p_1, p_2, \dots, p_n) \leqslant f(x_1', y, u, -p_1, p_2, \dots, p_n). \end{cases} \tag{2.3}$$

Observe that if $p_1 < 0$ then, by Lipschitz continuity, $\exists\, C \geqslant 0$ such that for $x_1 < x_1', x_1 + x_1' < 0$,

(1) A more general form, for fully nonlinear elliptic equations was given in Theorem 2.1'.

$$\begin{aligned} &f(x_1', y, u, -p_1, p_2, \dots, p_n) - f(x_1, y, u, p_1, p_2, \dots, p_n) \\ &\geqslant f(x_1', y, u, 0, p_2, \dots, p_n) - f(x_1, y, u, 0, p_2, \dots, p_n) + Cp_1 \\ &\geqslant Cp_1 \text{ by (2.3).} \end{aligned}$$

It follows that there is an L^∞ function $\beta \geqslant 0$, such that for $x_1 < x_1'$, $x_1 + x_1' < 0$, and all p_1,

$$f(x_1', y, u, -p_1, p_2, \dots, p_n) - f(x_1, y, u, p_1, p_2, \dots, p_n) \geqslant \beta p_1. \tag{2.4}$$

Here β depends on x_1, x_1', p, etc.

THEOREM 2.1. *With u and f as above, assume*

$$u(-a, y) \leqslant u(x_1, y) \text{ for } -a < x_1 < a, y \in \omega \tag{2.5}$$

and, $\forall x_1$ *in* $(-a, a)$, $\exists y \in \omega$ *such that strict inequality holds.*

In addition we assume that for $y \in \partial\omega$, $-a < x_1 < x_1' < a$

$$u(x_1, y) \leqslant u(x_1', y) \text{ provided } x_1 + x_1' < 0. \tag{2.6}$$

Then, for $-a < x_1 < x_1'$, $x_1 + x_1' < 0$, $y \in \omega$ *we have*

$$u_1(x_1, y) > 0, \ u(x_1, y) < u(x_1', y). \tag{2.7}$$

Furthermore if $u_1(0, y) = 0$ *for some* $y \in \omega$ *then u is a symmetric function of* x_1.

REMARK 2.1. We have stated the theorem in case $n > 1$, but of course it holds also for $n = 1$ – simply ignore the boundary conditions on $\partial\omega$. Here is a simple example in case $n = 1$ showing that the condition (2.3) on f cannot be dropped: On the interval $\Omega = (-a, a)$, a large, the conclusion of Theorem 2.1 does not hold for

$$u = (x + a)e^{-x}.$$

It satisfies

$$\ddot{u} + 2\dot{u} + u = 0,$$

and condition (2.3) does not hold. Furthermore if u is any C^3 function on $[-a, a]$ with $u(-a) < u(x) < u(a)$ which is not monotone, it still satisfies a differential equation $\ddot{u} + f(x) = 0$ where f is in fact $-\ddot{u}(x)$.

Observe also (see (2.5)) that on the interval $|x| < a = 3\pi/2$ the function

$$u = 1 - \sin x \text{ satisfies} \tag{2.8}$$

$$\ddot{u} + u - 1 = 0,$$

and satisfies $u(-a) < u(x) < u(a)$ for $|x| < a$ except at $x = \pm\pi/2$, and (2.7) does not hold for u. For $n = 2$ we may take $a = 3\pi/2$, $\omega = (0, \pi)$ and $u = 1 - \sin x \sin y$, which satisfies $\Delta u + 2u - 2 = 0$. Condition (2.6) holds as does (2.5) except at $x = \pi/2$, and (2.7) does not hold.

The proof of the Theorem is similar to those of Theorems 2.1 and 3.1 in [4] but differs in some essential details. In particular we do not have the analogue of Lemma 2.1 of [4].

We will use similar notation: For any λ in $-a < \lambda \leqslant 0$ let $\Sigma(\lambda)$ denote the finite cylinder

$$\Sigma(\lambda) = \{(x_1, y) \in \Omega; -a < x_1 < \lambda\}.$$

$\Sigma'(\lambda)$ is the reflection of $\Sigma(\lambda)$ in the plane

$$T_\lambda = \{x_1 = \lambda\};$$

the reflection of $x = (x_1, y)$ in the plane T_λ is the point $x^\lambda = (2\lambda - x_1, y)$.

Proof of Theorem 2.1 : In $\Sigma(\lambda)$ consider the functions

$$v(x) = u(x^\lambda) = u(2\lambda - x_1, y), \text{ and } w(x) = w_\lambda(x) = v(x) - u(x). \tag{2.9}$$

We will prove that for every λ in $(-a, 0)$,

$$w(x) > 0 \text{ in } \Sigma(\lambda) \tag{2.10}$$

$$-2u_1 = \partial_1 w < 0 \text{ on } T_\lambda \cap \Omega. \tag{2.11}$$

These yield (2.7). We use the method of moving planes; it consists of two steps:
(I) Initial step: prove (2.10), (2.11) for $0 < a + \lambda$ small.
(II) Continuation: prove the inequalities for all λ in $(-a, 0)$.

(I) Here we use condition (2.4) to derive a parabolic differential inequality for w in $\Sigma(\lambda)$. There v satisfies

$$\begin{aligned} -\Delta v &= f(x^\lambda, v, -v_1, \nabla_y v). \\ &\geqslant f(x, v, v_1, \nabla_y v) + \beta v_1 \end{aligned}$$

with $\beta \in L^\infty$, $\beta(x, \lambda) \geqslant 0$, by (2.4). Hence w satisfies

$$\begin{aligned} -\Delta w &= f(x^\lambda, v, -v_1, \nabla_y v) - f(x, u, \nabla u) \\ &\geqslant f(x, v, \nabla v) - f(x, u, \nabla u) + \beta v_1 . \end{aligned}$$

Since $f(x, u, p)$ is Lipschitz continuous in (u, p) it follows that for suitable bounded functions b_j, c,

$$\Delta w + \sum_1^n b_j w_j + cw + \beta v_1 \leqslant 0.$$

But

$$\frac{\partial}{\partial\lambda} w = 2u_1(x^\lambda, y) = -2v_1(x).$$

Thus we obtain a (degenerate) *parabolic* inequality for w as a function of x and λ:

$$\Delta w + b_j w_j + cw - \frac{\beta}{2}\frac{\partial}{\partial\lambda} w \leqslant 0. \tag{2.12}$$

It holds in a region V in (x, λ) space:

$$V = \{(x_1, y, \lambda); -a < x_1 < \lambda < \lambda_0, y \in \omega\} \tag{2.12'}$$

Except on the top part, $\lambda = \lambda_0$, we have $w \geqq 0$ on ∂V. Indeed on the boundary where $x_1 = \lambda$ we have $w = 0$, and where $x_1 = -a$ we have $w(x, \lambda) \geqq 0$ by (2.5). For $0 < a + \lambda_0$ small, the width of this region in the x_1 direction is small, namely $a + \lambda_0$. We may therefore apply Proposition 1.1 and conclude that

$$w \geqq 0 \text{ in } \Sigma(\lambda),$$

and hence

$$-2u_1 = w_1 \leqslant 0 \text{ on } T_\lambda \cap \Omega.$$

this is true for every λ close to $-a$. So $u_1 \geqslant 0$ for x_1 close to $-a$.

To finish Step (I) we use (2.3) and derive an *elliptic* inequality for w in $\Sigma(\lambda)$: As before we have

$$\begin{aligned}
-\Delta w &= f(x^\lambda, v, -v_1, \nabla_y v) - f(x, u, u_1, \nabla_y u) \\
&= f(x^\lambda, v, -v_1, \nabla_y v) - f(x^\lambda, u, -u_1, \nabla_y u) \\
&\quad + f(x^\lambda, u, -u_1, \nabla_y u) - f(x, u, u_1, \nabla_y u) \\
&\geqslant f(x^\lambda, v, -v_1, \nabla_y v) - f(x^\lambda, u, -u_1, \nabla_y u)
\end{aligned}$$

by (2.3) (we have shown that $u_1 \geqslant 0$ in $\Sigma(\lambda)$). Thus

$$\Delta w + \Sigma\ b_j w_j + cw \leqslant 0 \tag{2.13}$$

for suitable bounded functions b_j, c.

Note that (2.13) holds in $\Sigma(\lambda)$, for any λ in $(-a, 0)$ provided we know $u_1 \geqslant 0$ in $\Sigma(\lambda)$.

Since $w \geqq 0$ in $\Sigma(\lambda)$ for $0 < a + \lambda$ small, we infer from the maximum principle and the Hopf lemma that (2.10) and (2.11) hold. In fact we have proved

LEMMA 2.1. *Assume that for some λ in $(-a, 0)$ we have*

(2.14) $$u_1(x) \geqslant 0, \; u(x) \lneqq u(x^\lambda) \text{ in } \Sigma(\lambda).$$

Then (2.10), (2.11) hold, i.e.

(2.15) $$u(x) < u(x^\lambda) \text{ in } \Sigma(\lambda) \text{ and } u_1(\lambda, y) > 0 \text{ for } y \in \omega.$$

Step (I) is finished. Turn to Step (II). Inequalities (2.14) hold for every λ in a maximal interval $(-a, \mu]$ in $(-a, 0]$. We will prove that $\mu = 0$. Suppose the contrary, that $\mu < 0$. By Lemma 2.1 we have (2.15) for $\lambda = \mu$.

Since μ is maximal, only two situations are possible. Case 1. There is a sequence of points x^i with $u_1(x^i) < 0$ and $x_1^i \searrow \mu$. Case 2. There are sequences $\lambda^i \searrow \mu$, and $x^i \in \Omega$ with $x_1^i < \lambda^i$, such that

(2.16) $$u(x^i) > u((x^i)^{\lambda^i}).$$

Consider first case 1. For a suitable subsequence we have $x^i \to x$ on T_μ. Because of (2.15), necessarily $x \in \partial\Omega$. We may suppose that the exterior unit normal to $\partial\Omega$ at x is $e_2 = (0, 1, 0, \ldots, 0)$. From (2.6) we have $u_1 \geqslant 0$ at x; by continuity, $u_1 = 0$ at x. Moving from x^i along a segment in the direction e_2, in a short distance we hit ∂S_a, where $u_1 \geqslant 0$. Consequently at some point on that segment we must have $u_{12} \geqslant 0$. So $u_{12}(x) \geqslant 0$. The function w defined in (2.9), for $\lambda = \mu$, has in $\overline{\Sigma(\mu)}$ a minimum, zero, at the corner point x. By Lemma S of [4] we have

$$(\partial_1 + \partial_2) w < 0 \text{ or } (\partial_1 + \partial_2)^2 w > 0 \text{ at } x.$$

i.e. $-2u_1 < 0$ or $-4u_{12} > 0$ at x. But both are impossible. Thus Case 1 cannot occur.

Consider case 2. We may suppose that in $\Sigma(\lambda^i)$, $w = w_{\lambda^i}$ assumes its minimum, which is negative, at x^i. So $\nabla w = 0$ and the Hessian matrix of spatial second derivatives of w, $\{w_{jk}\} \geqslant 0$ there. For a subsequence we have $x^i \to \bar{x} \in \overline{\Sigma(\mu)}$. In the limit $u(\bar{x}) \geqslant u(\bar{x}^\mu)$ so we must have equality. From now on w refers to w_μ. By continuity, at x,

(2.17) $$w = 0, \; \nabla w = 0 \text{ and } w_{jk} \geqslant 0.$$

From (2.13) it follows that $\{w_{jk}\} = 0$ there. By the theorem of the mean there is a point z^i in the interval joining x^i to $(x^i)^{\lambda^i}$ where $u_1(z^i) < 0$. In view of (2.15) and the fact that Case 1 is impossible it follows that $\bar{x} \notin T_\mu$. Thus $\bar{x} = (\bar{x}_1, \bar{y})$ with $-a \leqslant \bar{x}_1 < \mu$. In $\overline{\Sigma(\mu)}$, w achieves its minimum, zero, at $\bar{x}$.

If $-a = \bar{x}_1$, $\bar{y} \in \omega$ then by Lemma H, $w_1(\bar{x}) > 0$, contradiction. Se we have $\bar{y} \in \partial\omega$ and $\bar{x}_1 > a$ or $\bar{x}_1 = a$. We may suppose that the exterior normal to $\partial\omega$ at $\bar{y}$ is $(1, 0, \ldots, 0)$. Then in the first case, $\bar{x}_1 > a$, Lemma H implies $w_2(\bar{x}) < 0$, contradiction. In the second case, $\bar{x}_1 = a$, Lemma S implies

$$(\partial_1 - \partial_2)^2 w(\bar{x}) > 0.$$

again a contradiction.

We have proved that $\mu = 0$. To complete the proof of Theorem 2.1 we must prove the last assertion. Suppose $u_1(0, y) = 0$ for some $y \in \omega$. For w defined in (2.9) with $\lambda = 0$ we have $w \geqslant 0$ in $\Sigma(0)$, and, as before, it satisfies an inequality of the form (2.13). It follows from the maximum principle that either $w \equiv 0$ or $w > 0$. In the latter case, by Lemma H, we would have $-2u_1 = w_1 < 0$ at $(0, y)$, a contradiction. So $w \equiv 0$, i.e. u is symmetric in x_1. Theorem 2.1 is proved.

REMARK 2.2. If we knew that $u_1 > 0$ near $x_1 = -a$ then Step (I) in the proof would be trivial.

As in [4], Theorem 2.1 yields immediately the

Proof of Theorem 1.1: By Theorem 2.1 we have $u_1 > 0$ for $x_1 < 0$ and

$$u(x) \leqslant u(x^\lambda) \text{ for } \lambda = 0.$$

But if we reflect the problem about the plane $x_1 = 0$, i.e. replace x_1 by $-x_1$, we obtain the same equation. Thus we may conclude that

$$u(x) \geqslant u(x^\lambda) \text{ for } \lambda = 0.$$

Hence u is symmetric in x_1. ∎

Next we will prove a monotonicity result for solutions of (2.2) in S_a under Neumann boundary conditions on $\partial\omega$.

THEOREM 2.2. *Assume the conditions of Theorem 2.1 but with (2.6) replaced by*

$$(\widetilde{2.6}) \qquad u_\nu(x_1, y) = 0 \text{ for } y \in \partial\omega.$$

where ν is the exterior unit normal to S_a at (x_1, y). Then for $-a < x_1 < x_1'$, $x_1 + x_1' < 0$ and $y \in \bar{\omega}$ we have

$$(2.18) \qquad u_1(x_1, y) > 0$$

$$(2.19) \qquad u(x_1, y) < u(x_1', y).$$

Furthermore if $u_1(0, y) = 0$ for some $y \in \bar{\omega}$ then u is a symmetric function of x_1.

The proof will rely on an extension of Proposition 1.1. in the appendix, Proposition A.1.

Proof: (a) We proceed as in the proof of Theorem 2.1 – with slight variations. Using (2.4) we first show that (2.10) and (2.11) hold for $0 < \lambda + a$ small. When using (2.4) we used Proposition 2.1. This was applied in the narrow region $-a < x_1 < \lambda < \lambda_0$, $y \in \omega$; we had $w \geqslant 0$ on its boundary (except on top) and concluded that $w \geqslant 0$ in $\Sigma(\lambda)$. In the present situation we have $w \geqslant 0$ on the left ($x_1 = -a$) and right boundaries of $\Sigma(\lambda)$ and $\partial_\nu w = 0$ on the remaining boundary for $\lambda < \lambda_0$. With the aid of Proposition A.1 in the appendix we find $w \geqslant 0$ in $\Sigma(\lambda)$ for λ close to $-a$. As before, using (2.13) we then find that $w > 0$ in $\Sigma(\lambda)$.

(b) We have

LEMMA 2.2. *Under the conditions of Lemma 2.1 we have (2.15) and*

$$u_1(\lambda, y) > 0 \quad \text{for } y \in \bar{\omega}. \tag{2.20}$$

Proof: As before, (2.20) holds for $y \in \omega$. Suppose $u_1(\lambda, y) = 0$ for some $y \in \partial\omega$. We may suppose the exterior normal ν there is e_2. Applying Lemma S to w_λ in $\Sigma(\lambda)$ at (λ, y) we infer that

$$-4u_{12} = (\partial_1 + \partial_2)^2 w > 0 \text{ there.}$$

But since $u_2 = u_\nu \equiv 0$ on a segment containing (λ, y) parallel to the x_1 axis we have $u_{12}(\lambda, y) = 0$. Contradiction. ∎

(c) With μ as in the proof of Theorem 2.1 we wish to show $\mu = 0$. Suppose $\mu < 0$. As before, we have the Cases 1 and 2. By Lemma $\widetilde{2.2}$, Case 1 cannot occur so we have case 2: $x^i \to \bar{x} \in \overline{\Sigma(\mu)}$; at $\bar{x}$, $w = 0$, $\nabla w = 0$ and $\{w_{jk}\} = 0$ for $w = w_\mu$. So $\bar{x} \in \partial\Sigma(\mu)$, and in view of (2.20), $-a \leqslant \bar{x}_1 < \mu$. If $-a < \bar{x}_1 < \mu$ then $\bar{y} \in \partial\omega$ and by Lemma H we find $\partial_\nu w(\bar{x}) < 0$ – contradiction. If $-a = \bar{x}_1$, and $y \in \omega$ we see again by Lemma H that $w_1(\bar{x}) > 0$ – contradiction. So $-a = \bar{x}_1$, $\bar{y} \in \partial\omega$. Using Lemma S there we obtain again a contradiction. Thus $\mu = 0$.

The last assertion of Theorem 2.1 holds as before. Theorem 2.2 is proved. ∎

Theorems 2.1 and 2.2 have immediate applications to infinite cylinders. Let S be the infinite cylinder $(-\infty, \infty) \times \omega$.

COROLLARY 2.1. *Let u be a C^2 solution of (2.2) in $\bar{S}$ with f satisfying condition (2.3). Assume that for some numbers $c_0 < c_1$, and $R > 0$,*

$$(2.21)\qquad \begin{cases} \overline{\lim}_{x_1 \to -\infty} u(x_1, y) \leqslant c_0 \textit{ uniformly for } y \in \bar{\omega} \\ u_1(x_1, y) > 0 \textit{ for } x_1 < -R \textit{ in } S \\ u(x_1, y) \geqslant c_1 \textit{ for } x_1 \geqslant -R \textit{ in } S. \end{cases}$$

(a) If everywhere on ∂S *we have* $u_1 \geqslant 0$, *then* $u_1 > 0$ *in* S.
(b) If everywhere on ∂S *we have* $u_\nu = 0$ *then* $u_1 > 0$ *in* $\bar{S}$.

Proof: Simply apply Theorems 2.1 and 2.2 and Remark 2.2, Lemma 2.2, in cylinders $\{-a < x_1 < b\} \times \omega$ for arbitrarily large a and b.

Extensions and applications of this kind of result are presented in our forthcoming paper [9]. In the next section we consider noncylindrical domains.

3. GENERAL SMOOTH DOMAINS

Here we extend the monotonicity results to more general domains. In addition, we present results on monotonicity in x_1 in the entire domain; see also the next section.

We will begin with an extension of Theorem 2.1 to general domains Ω. For convenience we will assume Ω has no corners: Ω is a bounded domain with smooth boundary and convex in the x_1 direction. We suppose

$$-a = \min\{x_1 ; x \in \bar{\Omega}\}.$$

Assume that

$$(3.1)\qquad (x_1, y) \in \Omega, x_1 < 0 \Rightarrow (x_1', y) \in \Omega \text{ for } x_1 < x_1' < -x_1 .$$

This implies that for $x \in \partial\Omega$, $x_1 < 0$, the unit exterior normal ν to $\partial\Omega$ at x has $\nu_1 \leqslant 0$.

In $\bar{\Omega}$ let u be a C^3 solution of (2.2) where f satisfies the conditions of Theorem 2.1. In particular

$$\text{for } x_1 < x_1', x_1 + x_1' < 0,$$

$$(3.2)\qquad f(x_1, y, u, p_1, p_2, \ldots, p_n) \leqslant f(x_1', y, u, -p_1, p_2, \ldots, p_n) \text{ if } p_1 > 0.$$

Corresponding to (2.5), (2.6) we make the following assumptions:

$$(3.3)\qquad \text{If } (x_1, y) \in \partial\Omega \text{ and } (x_1', y) \in \Omega, x_1 < x_1' \text{ then } u(x_1, y) < u(x_1', y)$$

$$(3.4)\qquad \begin{cases} \text{If } (x_1, y), (x_1', y) \in \partial\Omega, x_1 < x_1', x_1 + x_1' < 0 \text{ then} \\ u(x_1, y) \leqslant u(x_1', y). \end{cases}$$

From conditions (3.3), (3.4) it follows that

(3.5) $u_1 \geqslant 0$ on $\partial\Omega \cap \{x_1 \leqslant 0\}$.

As before we define for $\lambda > -a$,

$$\Sigma(\lambda) = \{x \in \Omega;\ x_1 < \lambda\}$$

$$\Sigma'(\lambda) = \text{the reflection of } \Sigma(\lambda) \text{ in the plane } T_\lambda = \{x_1 = \lambda\}.$$

THEOREM 3.1. *Assume the conditions above, i.e., (3.1)-(3.4). Then if* (x_1, y), $(x_1', y) \in \Omega$, $x_1 < x_1'$, $x_1 + x_1' < 0$ *we have*

$$u_1(x_1, y) > 0, u(x_1, y) < u(x_1', y). \tag{3.6}$$

Furthermore if $u_1 = 0$ *for some points on* $T_0 \cap \Omega$ *then* u *is a symmetric function of* x_1.

Proof: (i) In $\Sigma(\lambda)$ we consider, as before, the functions $v(x) = u(x^\lambda)$ and $w = w_\lambda$ given by (2.9), and we wish to show that for $-a < \lambda < 0$, inequalities (2.10) and (2.11) hold. As before, using (3.2) one derives (2.12), and establishes (2.10), (2.11) for λ close to $-a$. Thus if (2.10), (2.11) hold for all λ in $(-a, \lambda_0)$, then (2.7) holds for $x_1 < \lambda_0$, $x_1 < x_1'$, $x_1 + x_1' < 2\lambda_0$.

(ii) Next we have

LEMMA 3.1. *Assume that for some* λ *in* $-a < \lambda < 0$,

$$u_1(x) \geqslant 0,\ u(x) \leqslant u(x^\lambda) \text{ in } \Sigma(\lambda). \tag{3.7}$$

Then

$$\begin{cases} u_1(x) > 0 \text{ for } x \in T_\lambda \cap \Omega, \\ u(x) < u(x^\lambda) \text{ in } \Sigma(\lambda). \end{cases} \tag{3.8}$$

Furthermore if $(\lambda, y) \in \partial\Omega$ *and* $\nu_1 > -1/2$ *there, then in a neighbourhood of* (λ, y) *in* Ω *we have*

$$u_1 > 0. \tag{3.9}$$

Proof: The proof of (3.8) is similar to that of (2.15) and will be omitted. Turn to (3.9); at (λ, y) we may suppose the unit normal $\nu = (\nu_1, \nu_2, 0, \ldots, 0)$. Since $\nu_1 > -1/2$ we see that at (λ, y) the two bounding surfaces of $\Sigma(\lambda)$, $\partial\Omega$ and T_λ, meet at an angle $\theta > \pi/3$ (θ is the angle in $\Sigma(\lambda)$).

In $\Sigma(\lambda)$ the function $w = w_\lambda$ is positive and satisfies (2.13). At (λ, y) it achieves its minimum. We may apply Lemma A.1 of [4] and infer that at (λ, y)

$$(\partial_1 + b\partial_2)w < 0 \text{ or } (\partial_1 + b\partial_2)^2 w > 0$$

for b large, i.e.

$$u_1 > 0 \text{ or } u_{12} < 0 \text{ at } (\lambda, y).$$

If $u_1 > 0$ at (λ, y) then we are through. Il $u_{12} < 0$ at (λ, y) then it is < 0 in a neighbourhood in Ω. But $u_1 \geqslant 0$ on $\partial\Omega$. It follows that $u_1 > 0$ in a neighbourhood of (λ, y) in Ω.

The lemma is proved. ∎

(iii) As before we have (3.7) for all λ in a maximal interval $(-a, \mu]$ in $(-a, 0]$, and we wish to prove $\mu = 0$. Suppose $\mu < 0$. By Lemma 3.1, inequalities (3.8) hold.

We must treat the same two cases as in the proof of Theorem 2.1. Consider first Case 1: we have a sequence x^i with $x_1^i \searrow \mu$ and $u_1(x^i) < 0$. We will show that for $\epsilon > 0$ sufficiently small, and for every λ in some interval $\mu < \lambda < \lambda_0$, the function $w_\lambda \geqslant 0$ in the region

$$\Omega_{\epsilon,\lambda} = \{x \in \Omega; \mu - \epsilon < x_1 < \lambda\}.$$

This implies that $u_1 \geqslant 0$ in $\Omega_{\epsilon,\lambda_0}$ — contradicting the assumption in case 1.

To show $w_\lambda(x) \geqslant 0$ in the region

$$\tilde{\Omega}_\epsilon = \{(x, \lambda); x \in \Omega, \mu - \epsilon < x_1 < \lambda, \mu < \lambda < \lambda_0\}$$

in (x, λ) space, we will use the parabolic inequality (2.12), which follows, as before, using (3.2). We have to check that $w_\lambda \geqslant 0$ at every point on $\partial\tilde{\Omega}_\epsilon$ except those on $\lambda = \lambda_0$. As in §2 it will then follow with the aid of Proposition 1.1 that $w_\lambda \geqslant 0$ in $\tilde{\Omega}_\epsilon$. We will show that

$$w_\lambda \geqslant 0 \text{ on } \partial\Omega_{\epsilon,\lambda} \setminus T_\lambda. \tag{3.10}$$

In order to establish (3.10) we divide the points on $\partial\Omega_{\epsilon,\lambda} \setminus T_\lambda$ into different classes.

The set of points K on $T_\mu \cap \partial\Omega$ at which $\nu_1 \geqslant -1/3$ is compact. Thus for some $\delta > 0$ we see from (3.9) that $u_1 \geqslant 0$ at every point in K_δ. Here

$$K_\rho := \{x \in \bar{\Omega}; \mathit{dist}\ (x, K) < \rho\}.$$

For $0 < \epsilon$, $\lambda - \mu$ small, at any point

$$(x_1, y) \in J := \{x \in \partial\Omega \setminus K_{\delta/2}; \mu - \epsilon < x_1 < \lambda\},$$

we have $\nu_1 \leqslant -1/4$ and $u(x_1', y) > u(x_1, y)$, if $(x_1', y) \in \Omega$ and $x_1' > x_1$ – by (3.3). In particular for $\lambda - \mu > 0$ but small,

$$w_\lambda(x) > 0 \text{ for } x \in J. \tag{3.11}$$

Consider the compact set

$$L = (T_{\mu-\epsilon} \cap \overline{\Omega}) \setminus K_{\delta/2}.$$

On L we have $w_\mu \geqslant c_1$, a small positive constant. Hence for $0 < \lambda - \mu$ small we also have

$$(3.11)' \qquad w_\lambda > 0 \text{ on } L.$$

We have established $w_\lambda > 0$ at all points of $\partial\Omega_{\epsilon,\lambda} \setminus T_\lambda$ except those in $K_{\delta/2}$. But for $0 < \epsilon$, $\lambda - \mu$ small, we see that if $x \in K_{\delta/2}$ and $\mu - \epsilon < x_1 < \lambda$, then the interval joining x to x^λ lies in K_δ and so $u_1 \geqslant 0$ on it. Hence $w_\lambda(x) \geqslant 0$. We have thus verified (3.10). Consequently $w_\lambda \geqslant 0$ in $\Omega_{\epsilon,\lambda}$ for $\mu < \lambda < \lambda_0$. This implies $u_1 \geqslant 0$ in $\Omega_{\epsilon,\lambda_0}$. Contradiction.

Consider now case 2. We have $\lambda^i \searrow \mu$, and a sequence $x^i \to \overline{x} \in \overline{\Sigma(\mu)}$ with $u(x^i) > u((x^i)^{\lambda^i})$, and $\nabla w_{\lambda^i} = 0$ at x^i. As before, since case 1 is impossible, $\overline{x} \in \partial\Omega$ and $\overline{x}_1 < \mu$. Also $w_\mu = 0$, $\nabla w_\mu = 0$ at $\overline{x}$. By Lemma H, $\partial_\nu w_\mu(\overline{x}) < 0$, for ν the exterior unit normal at $\overline{x}$-contradiction.

Thus $\mu = 0$ and the proof of Theorem 3.1 is complete except for the last assertion. That is proved just as in Theorem 2.1. ∎

THEOREM 3.1$'$. *Assume the conditions of Theorem 3.1 and assume that Ω is symmetric in x_1 and that the boundary values of u are symmetric in x_1. Assume also f is symmetric in (x_1, p_1). Then u is symmetric in x_1.*

It is reasonable to ask if one can prove a monotonicity (in x_1) result in all of Ω. Here is such a result. For convenience we suppose again that $\partial\Omega$ is smooth.

THEOREM 3.2. *Let Ω be a bounded domain in R^n with smooth boundary and which is convex in the x_1 direction. In Ω let u be a C^3 solution of (2.2) where f satisfies (3.2) $f(x_1, y, u, p_1, \ldots, p_n) \leqslant f(x_1', y, u, -p_1, p_2, \ldots, p_n)$ if $p_1 \geqslant 0$ for all $(x_1, y), (x_1', y)$ in Ω with $x_1 < x_1'$. Assume that on every interval in Ω parallel to the x_1-axis, with end points x^0, $x^0 + te_1$ on $\partial\Omega$, $t > 0$, we have*

$$(3.12) \qquad u(x^0) < u(x^0 + se_1) < u(x^0 + te_1) \qquad 0 < s < t$$

Assume also (A): If $\partial\Omega$ contains a segment parallel to the x_1 axis on which u is constant then ν, the unit exterior normal to $\partial\Omega$, is also constant on the segment. (This condition is automatic if $n = 2$). Assume also that at any boundary point of Ω, where $\nu_1 = 0$, we have

$$(3.12)' \qquad u_1 \geqslant 0.$$

Then

(3.13) $u_1 > 0$ in Ω.

From (3.12) and (3.12), it follows that $u_1 \geqslant 0$ everywhere on $\partial\Omega$ – so one might expect (3.13) to follow with the aid of the maximum principle. But the principle is not immediately applicable.

Proof: We suppose that min x_1 and max x_1 in $\overline{\Omega}$ are $-a$ and 0, and we define $\Sigma(\lambda)$ and its reflection $\Sigma'(\lambda)$ as above. For λ greater than but close to $-a$ we have $\Sigma'(\lambda) \subset \Omega$, but this will no longer be the case as we increase λ. In place of $\Sigma(\lambda)$ we will work with

$$\Omega(\lambda) = \{x \in \Sigma(\lambda);\ x^\lambda \in \Omega\}.$$

In $\Omega(\lambda)$ we consider the function $w_\lambda(x)$ of (2.9). From our conditions it follows that $w = w_\lambda \geqq 0$ on $\partial\Omega(\lambda)$. To prove the theorem we will prove the analogues of (2.10), (2.11): for $-a < \lambda < 0$.

(3.14) $w_\lambda(x) > 0$ in $\Omega(\lambda)$

(3.15) $w_1 < 0$ on $T_\lambda \cap \Omega$.

The arguments will be similar to those used above.

(i) Just as before we find that (3.14) and (3.15) hold for λ close to $-a$.

(ii) The analogue of Lemma 3.1 (and proved in the same way) is

LEMMA 3.1′. *Assume that for some λ in $-a < \lambda < 0$,*

(3.16) $u_1(x) \geqslant 0,\ u(x) \leqslant u(x^\lambda)$ in $\Omega(\lambda)$.

Then

$u_1(x) > 0$ for $x \in T_\lambda \cap \Omega$,

(3.17) $u(x) < u(x^\lambda)$ in $\Omega(\lambda)$.

Furthermore if $(\lambda, y) \in \partial\Omega$ and $|\nu_1| < 1/2$ there, then in a neighbourhood of (λ, y) in Ω we have

(3.18) $u_1 > 0$.

(iii) We have (3.16) for all λ in a maximal interval $(-a, \mu]$ in $(-a, 0]$, and we wish to prove $\mu = 0$. As before, suppose $\mu < 0$. By Lemma 3.1′, inequalities (3.17) hold for $\lambda = \mu$. As usual we have two cases to exclude.

Case 1: There is a sequence x^i with $x_1^i \searrow \mu$ and $u_1(x^i) < 0$. We will show that for $\epsilon > 0$, sufficiently small, and for every λ in some interval $\mu < \lambda < \lambda_0$ the function $w_\lambda \geqslant 0$ in the region (which may have infinitely many components):

$$\Omega(\epsilon, \lambda) = \{x \in \Omega(\lambda); \mu - \epsilon < x_1 < \lambda\},$$

This implies that $u_1 \geqslant 0$ in $\cup_{\mu \leqslant \lambda < \lambda_0} \Omega(\epsilon, \lambda)$, which contains a neighbourhood of $T_\mu \cap \Omega$, so that case 1 cannot hold.

As in the proof of Theorem 3.1 we wish to verify that

$$w_\lambda \geqslant 0 \text{ on } \partial\Omega(\epsilon, \lambda). \tag{3.19}$$

On T_λ we have $w_\lambda = 0$. As before we divide the points on $\partial\Omega(\epsilon, \lambda) \setminus T_\lambda$ into different classes.

The set K of points on $T_\mu \cap \partial\Omega$ at which $|\nu| \leqslant 1/3$, is compact. So for some $\delta > 0$ we see from (3.9) that $u_1 \geqslant 0$ at every point in K_δ. For ϵ, $\lambda - \mu$ small, on

$$J := \{x \in \partial\Omega \setminus K_{\delta/2}; \mu - \epsilon < x_1 < \lambda \text{ and } \nu_1 < 0\}$$

we have $\nu_1 \leqslant -1/4$. If $(x_1, y) \in J$, $(x_1', y) \in \Omega$, $x_1' > x_1$ then $u(x_1', y) > u(x_1, y)$.

Thus $w_\lambda(x) > 0$ for $x \in J$. Let M be the set of points $x \in (\partial\Omega(\epsilon, \lambda) \setminus K_{\delta/2})$ with $\mu - \epsilon < x_1 < \lambda$ such that $x^\lambda \in \partial\Omega$. Then $\nu_1(x^\lambda) \geqslant 1/4$ and we find from (3.12) that

$$w_\lambda(x) > 0 \text{ for } x \in M.$$

On the compact set

$$L = (T_{\mu-\epsilon} \cap \partial\Omega(\epsilon, \lambda)) \setminus K_{\delta/2}$$

we have $w_\mu > c_1 > 0$. So for $0 < \lambda - \mu$ small we also have $w_\lambda > 0$ on L. Thus $w_\lambda > 0$ at all points of $\partial\Omega(\epsilon, \lambda) \setminus T_\lambda$ except possibly those in $K_{\delta/2}$. But as before we see that $w_\lambda(x) \geqslant 0$ there. So (3.19) is verified and case 1 is impossible.

Case 2. There is a sequence $\lambda^i \searrow \mu$ and a sequence of points $x^i \in \Omega(\lambda^i)$, with $u(x^i) > u((x^i)^{\lambda^i})$. So $u(\bar{x}) \geqslant u(\bar{x}^\mu)$. As before, since case 1 is impossible, it follows that $\bar{x}_1 < \mu$ and $\bar{x} \in \overline{\Omega(\mu)}$. Also $w_\mu(\bar{x}) = 0$. It follows from (3.12) and (3.12)$'$ that the straight segment σ joining $\bar{x}$ and $\bar{x}^\mu$ belongs entirely to $\partial\Omega$, and $u =$ constant on σ. Now if $n > 2$, at $\bar{x}$, the boundary of $\Omega(\mu)$ may have a sharp corner. So we cannot apply Lemma H. For that reason we assumed condition (A). Because of that condition, we *may* apply Lemma H in $\Omega(\mu)$ at $\bar{x}$, and conclude that $\partial_\nu w < 0$. We may then proceed as in the proof of Theorem 3.1 and we see that Case 2 is impossible. Thus $\mu = 0$ and the proof of Theorem 3.2 is complete. ∎

REMARK 3.1. In the theorem, if Ω is symmetric in x_1 about $x_1 = \bar{a}/2$ then Hypothesis (A) is automatically satisfied.

Can we drop Hypothesis (A) in general?

A monotonicity result in a full cylinder $\Omega = S_a$ may be proved in a similar way. The details are simpler and we merely state the result.

THEOREM 3.3. *Let u be a C^2 solution in $\bar{\Omega}$ of (2.2) where f satisfies (3.2) for $x_1 < x_1'$. Assume*

$$(3.20)\qquad \begin{cases} u(-a, y) \leqslant u(x_1, y) \leqslant u(a, y) \text{ for } -a < x_1 < a, y \in \omega, \\ \text{and } \forall x_1 \text{ in } -a < x_1 < a, \ \exists y, y' \in \omega \text{ such that} \\ u(-a, \ y) < u(x_1, y), u(x_1, y') < u(a, y'). \end{cases}$$

Assume also

$$(3.21)\qquad u_1(x_1, y) \geqslant 0 \text{ for } y \in \partial\omega.$$

Then

$$(3.22)\qquad u_1 > 0 \text{ in } \Omega.$$

Berestycki and Pacella [2] adapted the method of moving planes to derive symmetry results in sector-like domains. Here too we may consider other geometries and derive other forms of Theorem 3.2. One may use the method of moving planes but not require them to be parallel while moving. As an example we present such a result in an angular sector.

In R^n let $(\rho, \theta_1, \ldots, \theta_{n-1})$ be polar coordinates, $\rho \geqslant 0$, $\theta_i \in [0, \pi]$ for $1 \leqslant \leqslant i \leqslant n-2$, $\theta_{n-1} \in [0, 2\pi)$. In R^n let Ω be a bounded domain whose closure does not touch the x_n – axis and which, in polar coordinates, is a product domain

$$\omega \times (0 < \theta_{n-1} < \alpha).$$

Here ω is a bounded domain in R^{n-1} with smooth boundary, and $\alpha < 2\pi$.

THEOREM 3.4. *Let $u \in C^2(\bar{\Omega})$ be a solution of*

$$\Delta u + f(\rho, \theta_1, \ldots, \theta_{n-1}, u, u_\rho, u_{\theta_1}, \ldots, u_{\theta_{n-1}}) = 0$$

with f continuous, and Lipschitz continuous in $(u, \nabla u)$. Assume (for convenience) the analogue of (3.2) (here $p = (p_1, \ldots, p_{n-1})$).

$$(3.23)\qquad \begin{aligned} & f(\rho, \theta_1, \theta_2, \ldots, \theta_{n-1}, u, u_\rho, p_1, \ldots, p_{n-1}) \leqslant \\ & \leqslant f(\rho, \theta_1 \ldots, \theta'_{n-1}, \theta_n, u, u_\rho, \ p_1, p_2, \ldots, -p_{n-1}) \\ & \text{if } \theta_{n-1} < \theta'_{n-1}, p_{n-1} \geqslant 0. \end{aligned}$$

Assume that for every fixed $\sigma = (\rho, \theta_1, \ldots, \theta_{n-2})$ (corresponding to a point in ω),

$$(3.24) \qquad \begin{cases} u(\sigma, 0) \leqslant u(\sigma, s) \leqslant u(\sigma, \alpha) \text{ for } 0 < s < \alpha \\ \text{and } \forall s \text{ in } 0 < s < \alpha, \exists\, \sigma, \sigma' \text{ such that } u(\sigma, 0) < u(\sigma, s) \\ u(\sigma', s) < u(\sigma', \alpha) \end{cases}$$

Assume also $u_{\theta_{n-1}} \geqslant 0$ *on* $\partial\Omega \cap \{0 < \theta_{n-1} < \alpha\}$. *Then* $u_{\theta_{n-1}} > 0$ in Ω.

The proof is, again, left to the reader.

L. Caffarelli has pointed out to us a much simpler argument to prove monotonicity in a full cylinder – and even uniqueness for solutions satisfying (roughly) (3.24). Here is such a result, and his argument.

THEOREM 3.5. *In the cylinder* $\Omega = S_a$ *let* u *be a* $C^2(\Omega) \cap C^1(\bar{\Omega})$ *solution of (2.2) with* f *continuous and Lipschitz continuous in* $(u, \nabla u)$. *Assume that* f *is nondecreasing in* x_1. *Assume*

$$(3.20)' \qquad u(-a, y) < u(x_1, y) < u(a, y) \text{ for } -a < x_1 < a,\ y \in \omega$$

and either

$$(3.21)' \qquad u \text{ is strictly increasing in } x_1, \text{ on } \partial\Omega$$

or the Neumann condition

$$(3.21)'' \qquad u_\nu = 0 \text{ for } -a < x_1 < a,\ y \in \partial\omega.$$

Then (a) $u_1 \geqslant 0$ *in* Ω. *(b) If* f *is also Lipschitz in* x_1 *then* $u_1 > 0$ *in* Ω. *(c) There is at most one solution with given Dirichlet data, or with Neumann conditions (3.21)″, satisfying all these conditions, i.e.* u *is unique.*

Proof: (a) In the region $\Sigma(\lambda)$, for $-a < \lambda < a$, consider the function $v(x) = u(x_1 + a - \lambda, y)$, and set $w = v(x) - u(x)$. w satisfies

$$\begin{aligned} 0 &= \Delta w + f(x_1 + a - \lambda, y, v, \nabla v) - f(x, u, \nabla u) = 0 \\ &\geqslant \Delta w + f(x, v, \nabla v) - f(x, u, \nabla u) \\ &= \Delta w + \Sigma\, b_j w_j + cw \end{aligned}$$

for suitable coefficients b_j, c, having L^∞ norms bounded by a constant independent of λ. Also $w \gneqq 0$ on $\partial\Sigma(\lambda)$, or in case of (3.21)″, on part of $\partial\Sigma(\lambda)$, with $w_\nu = 0$ on the remaining part. We use the «sliding domain method»: For $0 < a + \lambda$ small we infer that $w > 0$ in $\Sigma(\lambda)$, i.e. v lies above u. Now increase λ. On $\partial\Sigma(\lambda)$ we always have $w = w_\lambda \gneqq 0$ for $\lambda < a$. So by the maximum principle we must continue to have $w > 0$ for every $\lambda < a$. This implies that u is increasing in x_1.

(b) Differentiating the equation (2.2) with respect to x_1 and applying the maximum principle again we find $u_1 > 0$.

(c) If $\underline{u}$ is another solution satisfying the same conditions, the same argument shows that $v(x) = u(x_1 + a - \lambda, y)$ is greater than $\underline{u}$ in $\Sigma(\lambda)$ for $0 < a + \lambda$ small. Increasing λ as before we find $\underline{u}(x) \geqslant u(x)$. Interchanging the roles of u and $\underline{u}$ we find the opposite inequality. ■

COROLLARY 3.1. *Let u, f satisfy the conditions of Theorem 3.5. Assume in addition that the boundary values of ϕ are odd in x_1. Assume also that f is odd in $(x_1, u, p_2, \ldots, p_n)$. Then u is odd in x_1.*

This follows, from the fact that $\underline{u} = -u(-x_1, y)$ is a solution satisfying the same conditions, and so equals u.

REMARK 3.2. The conditions in Theorems 3.3 and 3.5 are slightly different. Neither implies the other. Here is an example of an equation where the conditions of Theorem 3.3 hold but not those of Theorem 3.5. On the interval $-1 \leqslant x \leqslant 1$ the function $u = x$ satisfies the equation $\ddot{u} + \min(0, x)\dot{u} + \max(0, x)\dot{u}^2 - u = 0$.

REMARK 3.3. In Theorems 3.1, $\widetilde{3.1}$ and 3.2 we assumed $u \in C^3(\overline{\Omega})$. Li, Cong Ming pointed out to us that with slight modifications the proofs work if $u \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. This is because in Lemma A.1 of [4], the last assertion holds if $u \in C^{r,\alpha}$ in $\overline{\Omega}$ near 0 for $r + \alpha > \pi/\theta_0$. See Lemma A.1 in this paper.

The argument used in the proof of Theorem 3.5 yields the following result.

THEOREM 3.6. *Let f be as in Theorem 3.5 and let u, $\underline{u} \in C^3(\overline{\Omega})$ be solutions of (2.2) satisfying*

$$\underline{u}(x_1, y) < u(a, y),\ \underline{u}(-a, y) < u(x_1, y) \tag{3.25}$$

for $-a < x_1 < a$, $y \in \omega$. Assume also either

$$\underline{u}(x_1, y) < u(x_1', y) \text{ for } -a \leqslant x_1 < x_1' \leqslant a,\ y \in \partial\omega, \tag{3.26}$$

or

$$\underline{u}_\nu = u_\nu = 0 \text{ for } -a < x_1 < a,\ y \in \partial\omega. \tag{3.27}$$

Then

$$\underline{u} \leqslant u \text{ in } \Omega.$$

4. MONOTONICITY IN THE WHOLE DOMAIN AND ANTISYMMETRY

Theorems 3.3 and 3.5 concerned monotonicity in cylinders. Corollary 3.1 applied the latter to prove antisymmetry. In this section we will present a stronger version of Theorem 3.5 as well as Cor. 3.1.

Consider again the finite cylinder $\Omega = S_a = (-a, a) \times \omega$, with $a > 0$; as before $\omega \subset R^{n-1}$ is a bounded domain with smooth boundary. Let $u \in C^2(\bar{\Omega})$ be a solution of

$$\Delta u + f(x, u, \nabla u) = 0 \text{ in } \Omega \tag{4.1}$$

$$u = \phi \text{ on } \partial\Omega. \tag{4.2}$$

Here ϕ is a continuous function on $\partial\Omega$ satisfying

$$\phi(x_1, y) \leqslant \phi(x_1', y) \text{ for } x_1 \leqslant x_1'. \tag{4.3}$$

The function $f(x, y, p)$ is continuous in all variables, locally Lipschitz in (u, p) and satisfies

$$f(x, u, p) \text{ is nondecreasing in } x_1 \text{ for } p_1 \geqslant 0. \tag{4.4}$$

THEOREM 4.1. *Let u be as above and assume conditions (4.3), (4.4). Assume also that u satisfies*

$$\begin{cases} \phi(-a, y) \leqslant u(x_1, y) \leqslant \phi(a, y) \textit{ for } -a < x_1 < a, y \in \omega, \\ \textit{and } \forall x_1 \textit{ in } (-a, a), \exists y \in \omega \textit{ such that} \\ \phi(-a, y) < u(x_1, y). \end{cases} \tag{4.5}$$

Then u is strictly increasing in x_1 in Ω. Furthermore it is unique, i.e., if $\underline{u}$ is another solution of (4.1), (4.2) satisfying (4.5) then $\underline{u} = u$.

COROLLARY 4.1. *Under the additional assumptions that $u_1 \in C^2(\Omega)$ and that f is locally Lipschitz in (x_1, u, p) we have the stronger conclusion: $u_1 > 0$ in Ω.*

COROLLARY 4.2 (ANTISYMMETRY). *Assume the conditions of Theorem 4.1 and assume in addition that ϕ is odd in x_1, on $\partial\Omega$, and that $f(x, u, p)$ is odd in $(x_1, u, p_2, \ldots, p_n)$. Then u is odd, i.e. antisymmetric in x_1:*

$$u(-x_1, y) = -u(x_1, y) \;\forall (x_1, y) \textit{ in } \Omega.$$

COROLLARY 4.3. *Let f be in Theorem 4.1 and satisfy, in addition, for constants $A < B$,*

$$f(x, u, 0) \geqslant 0 \quad \textit{if } u \leqslant A, \; f(x, u, 0) \leqslant 0 \textit{ if } u \geqslant B.$$

Suppose u *and* $\underline{u} \in C^2(\bar{\Omega})$ *are solutions of (4.1), (4.2) where* ϕ *satisfies (4.3) and*

$$\phi \equiv A \text{ on } x_1 = -a, \ \phi \equiv B \text{ on } x_1 = a.$$

Then $u \equiv \underline{u}$, *i.e. the solution is unique, and* u *is strictly increasing in* x_1 *in* Ω.

Proof: This follows from the Theorem if we can show that

$$A < u, \underline{u} < B \text{ in } \Omega.$$

We will just show $u < B$; the other inequalities are proved in the same way. If $M = \max u > B$ then in a neighbourhood of a point where $u = M$ we have

$$\Delta u + f(x, u, p) - f(x, u, 0) \geqslant 0.$$

Using the Lipschitz continuity of f in p we find in that neighbourhood,

$$\Delta u + \Sigma b_i u_i \geqslant 0,$$

with $b_i \in L^\infty$. By the maximum principle, the set of points where $u = M$ is open. By continuity it is closed – and so all of $\bar{\Omega}$. Impossible. Next, if $u = B$ at some point in Ω then in Ω we have

$$\Delta(u - B) + f(x, u, p) - f(x, B, 0) \geqslant 0.$$

Using Lipschitz continuity in (u, p) we find

$$\Delta(u - B) + b_j(u - B)_j + c(u - B) \geqslant 0 \text{ in } \Omega.$$

Here the b_j and c are bounded measurable functions. But $u - B \leqslant 0$ in Ω, with equality holding at some point in Ω. The maximum principle implies $u \equiv B$. Impossible.

Even for $n = 1$ the result seems new – even for the simple equation

$$\ddot{u} + b(x)\dot{u} + f(u) = 0,$$

with b nondecreasing in x, and $g(u) \geqslant 0$ for $u \leqslant A, g(u) \leqslant 0$ for $u \geqslant B$. Li, Cong-Ming has constructed an example with b decreasing for which there is nonuniqueness. ∎

REMARK 4.1. The condition (4.5) cannot be dropped. For example, in R^2, with $\omega = (0, \pi)$, $\Omega = (-\pi/2, \pi/2) \times (0, \pi)$, the positive eigenfunction $\tilde{u} = \sin y \cos x_1$ of

$$\Delta\tilde{u} + 2\tilde{u} = 0 \text{ in } \Omega$$

$$\tilde{u} = 0 \text{ on } \partial\Omega$$

is not antisymmetric in x_1. Using this example one may in fact obtain a more interesting one: a solution u of the same equation which is not antisymmetric in x_1, and with boundary values ϕ which are strictly increasing (and odd) in x_1. Namely, let v be the solution of

$$\Delta v + 2v = 0 \text{ in } \Omega$$

$$v = \tilde{\phi} \text{ on } \partial\Omega$$

where $\tilde{\phi}$ is odd and strictly increasing in x_1 – it is easily verified that this is always solvable. Then take $u = \tilde{u} + \epsilon v$.

The proof of Theorem 4.1 which we will present in an extension of that of Caffarelli's of Theorem 3.5. We have another proof of Cor. 4.2 which is somewhat surprising in that it makes use of the method of moving planes and reflection. However in the region $\Sigma(\lambda)$, in place of the function w defined in (2.9), one works with the function $\tilde{w} = u(x) + u(x^\lambda)$. It satisfies Neumann boundary data on $x_1 = \lambda$. One proves that $\tilde{w} \leqslant 0$ in $\Sigma(\lambda)$.

Suppose $\underline{u}$ and u are solution satisfying all the conditions of the Theorem. We will show that in our usual $\Sigma(\lambda)$, for $-a < \lambda < a$, the (new) function

$$w(x) := u(x + (a - \lambda)e_1) - \underline{u}(x) > 0. \tag{4.6}$$

If we take $\underline{u} = u$ we infer that u is strictly increasing in x_1. If we set $\lambda = a$ we see that $u \geqslant \underline{u}$ in Ω. Interchanging the roles of u and $\underline{u}$ it follows that $u = \underline{u}$. Thus (4.6) yields Theorem 4.1.

To prove (4.6) we will derive a parabolic inequality for w of the form (2.12). But then we have need of parabolic analogues of Lemmas H and S. So we present forms of these which will suffice for our purposes. We recall from section 1 that V is a bounded domain in R^{n+1} lying in $t < T$.

Hypotheses:

1) Here w is a solution in $\overline{V}$ (see section 1) of (1.9):

$$(L - \beta\partial_t)w = a_{ij}(x, t)w_{x_i x_j} + b_i(x, t)w_{x_i} + c(x, t)w - \beta(x, t)w_t \geqslant 0 \tag{4.7}$$

where the a_{ij} are continuous and satisfy (1.8), and the other coefficients are in L^∞, also $\beta \geqslant 0$, w is supposed to have continuous second derivatives in the space variables x, and continuous time derivative $\partial_t w$ in $\overline{V}$.

2) $V_\tau = V \cap \{t = \tau\}$ $\forall \tau < T$ is connected, and V_T is connected. Here V_T consists of points (x, T) such that the lower open half of some ball with centre (x, T) is in V. Set

$$V \cup V_T = \tilde{V}.$$

In the following, P denotes a paraboloid

$$P = \{(x, t); t - T + \delta > | x - x^0 |^2\}, \quad \delta > 0.$$

for which the parabolic cap

(4.8) $\qquad Q = P \cap \{t \leqslant T\}$ lies in $\tilde{V}$.

We also consider parabolic caps with T replaced by some other value.

Here is a parabolic analogue of Lemma H (for $\beta > 0$ see [7], chapt. 3, section 3).

LEMMA 4.1. *($\tilde{H}$) Let V, w and Q be as above. Suppose $w < 0$ in Q and equals zero at a point $(\bar{x}, T) \in \partial Q \cap \partial P$. Then $w_\nu > 0$ there, where ν denotes any spatial outer direction to the sphere $| x - x^0 |^2 = \delta$ in the plane $t = T$.*

Using this one easily establishes

LEMMA 4.2. *Assume hypotheses 1), 2), and that $w \leqslant 0$ in V. Suppose $w < 0$ at some point $(x^0, t^0) \in \tilde{V}$. Then $w < 0$ on all of V_{t^0}.*

Next an analogue of Lemma S.

LEMMA 4.3 ($\tilde{\text{S}}$). *Assume hypotheses 1), 2) with $w \leqslant 0$ in $\tilde{V}$ and $w = 0$ at a point $(\bar{x}, T)$ on ∂V_T. Suppose that near $(\bar{x}, T)$, $\partial V \setminus V_T$ consists of two transversally intersecting C^2 hypersurfaces $\{\rho = 0\}$ and $\{\sigma = 0\}$, with $\rho, \sigma < 0$ in V and $\nabla_x \rho$, $\nabla_x \sigma$ linearly independent at $(\bar{x}, T)$. Assume that at $(\bar{x}, T)$;*

(4.9) $\qquad a_{ij} \rho_{x_i} \sigma_{x_j} = 0$

and assume that there exists a C^2 curve $\mathscr{C}$ of the form $(\xi(t), t)$ for $T - \epsilon < t \leqslant T$, lying in $\{\rho = \sigma = 0\}$ such that on $\{\rho = \sigma = 0\}$ we have

(4.10) $\qquad a_{ij} \rho_{x_i} \sigma_{x_j} \geqslant - C$ *(distance to $\mathscr{C}$)^2*

Conclusion: For any spatial outer direction ν at $(\bar{x}, T)$ (i.e. $\nabla_x \rho \cdot \nu, \nabla_x \sigma \cdot \nu > 0$) either

(4.11) $\qquad \partial_\nu w > 0 \quad$ or $\quad \partial_\nu^2 w < 0$ *at $(\bar{x}, T)$.*

We also have

LEMMA 4.4. *Assume the conditions of Lemma 4.3 except that, in place of (4.9) we have*

(4.9)$'$ $\qquad a_{ij} \rho_{x_i} \sigma_{x_j} > 0$ *at $(\bar{x}, T)$.*

Then

(4.12) $\partial_\nu w(\bar{x}, T) > 0.$

For the convenience of the reader, proofs of these lemmas will be sketched in the Appendix. See also Lemma A.1 there, a parabolic form of Lemma A.1 of [4].

Proof of Theorem 4.1: As indicated earlier, the theorem follows once (4.6) is established, i.e.,

(4.6)′ $$w(x, \lambda) = w(x) = v(x) - \underline{u}(x) > 0 \text{ in } \Sigma(\lambda), \text{ for } -a < \lambda < a.$$

Here

(4.13) $$v(x) = u(x + (a - \lambda)e_1) \quad \text{in} \quad \Sigma(\lambda).$$

We will prove (4.6)′ by deriving a parabolic differential inequality for w and then using the maximum principle. We have

$$\begin{aligned} 0 &= \Delta w + f(x_1 + a - \lambda, y, v, \nabla v) - f(x, \underline{u}, \nabla \underline{u}) \\ &= \Delta w + f(x, v, \nabla v) - f(x, \underline{u}, \nabla \underline{u}) \\ &+ f(x_1 + a - \lambda, y, v, \nabla v) - f(x, v, \nabla v). \end{aligned}$$

By condition (4.4) we see that

$$I = f(x_1 + a - \lambda, y, v, \nabla v) - f(x, v, \nabla v)$$

satisfies $I \geqslant 0$, if $v_1 \geqslant 0$ while, by Lipschitz continuity, $I \geqslant Cv_1$ for some constant $C > 0$, if $v_1 < 0$. Thus $I \geqslant \beta v_1$ where $\beta \geqslant 0$ is in L^∞. Hence

$$0 \geqslant \Delta w + f(x, v, \nabla v) - f(x, \underline{u}, \nabla \underline{u}) + \beta v_1$$

or

(4.14) $$0 \geqslant \Delta w + b_j w_j + cw - \beta \partial_\lambda w,$$

by the integral theorem of the mean and the identity $v_1 = -\partial_\lambda w$. This is our parabolic inequality. It holds in the region U in (x, λ) space:

$$U = \{(x, \lambda); x \in \Omega, -a < x_1 < \lambda, -a < \lambda < a\}$$

On the «spatial» boundary of U, i.e. the part of the boundary lying in $\lambda < a$ we have by (4.3) and (4.5), $w \geqslant 0$. In fact if we set

(4.15) $$U_t = U \cap \{\lambda = t\}, \quad -a < t < a,$$

then

(4.16) $$w \gneqq 0 \text{ on } \partial U_t.$$

Note that $U_t = \Sigma(t)$ is connected.

For $0 < a + \lambda$ small we may apply Proposition 1.1 and infer that

$$w(x, \lambda) \geqslant 0.$$

Since $U_\lambda = \Sigma(\lambda)$ is connected it follows from (4.16) and Lemma 4.2 that $w(x, \lambda) > 0$ for $x \in \Sigma(\lambda)$.

In $(-a < \lambda < a)$ there is a maximal open interval $(-a < \lambda < \mu)$ for which the inequality

$$w(x, \lambda) \geqslant 0 \quad \forall x \in \Sigma(\lambda) \tag{4.17}$$

holds. We wish to show that $\mu = a$. Suppose $\mu < a$ – we will obtain a contradiction. By continuity we have $w(x, \mu) \geqslant 0$ for $x \in \Sigma(\mu)$, and as before we infer that

$$w(x, \mu) > 0 \quad \text{for} \quad x \in \Sigma(\mu). \tag{4.17'}$$

By definition of μ there is a sequence $\lambda^i \searrow \mu$ and points $x^i \in \Sigma(\lambda^i)$ such that

$$\underline{u}(x^i) > u(x_1^i + a - \lambda^i, y^i) = u(\tilde{x}^i). \tag{4.18}$$

Here we have set $x^i + (a - \lambda^i)e_1 = \tilde{x}^i$. We may suppose that in $\overline{\Sigma(\lambda^i)}$, w has its (negative) minimum at x^i. So $\nabla_x w = 0, \{w_{jk}\} \geqslant 0$ there. A suitable subsequence x^i converges to a point $\bar{x}$ in $\overline{\Sigma(\mu)}$, with $\tilde{x}^i \to \bar{x} + (a - \mu)e_1 = \bar{\bar{x}}$. Hence

$$w(\bar{x}, \mu) = 0 \tag{4.19}$$

and so $\bar{x} \in \partial\Sigma(\mu)$. At $(\bar{x}, \mu)$, $\nabla_x w = 0, \{w_{jk}\} \geqslant 0$.

In (x, λ) space we set

$$V = U \cap \{\lambda < \mu\}$$

Several cases can occur and each has to be treated. In the following we usually write $w(x)$ to represent $w(x, \mu)$.

Case 1. $-a < \bar{x}_1 < \mu$. We may suppose e_2 is exterior normal to $\partial\Omega$ at $\bar{x}$. In V we may apply Lemma $\tilde{\text{H}}$ (i.e. Lemma 4.1) and conclude that

$$w_2(\bar{x}, \mu) < 0, \text{ contradiction.} \tag{4.20}$$

Case 2. $-a = \bar{x}_1$, $\bar{y} \in \omega$. Applying Lemma $\tilde{\text{H}}$ again we see that $w_1(\bar{x}) > 0$ – again a contradiction.

Case 3. $\bar{x} = (\mu, \bar{y})$, $\bar{y} \in \omega$. By Lemma $\tilde{\text{H}}$, $w_1(\bar{x}) < 0$ – contradiction.

Case 4. $\bar{x} = (-a, \bar{y})$, $\bar{y} \in \partial\omega$. This is treated as in the proof of Theorem 2.1. We know

$$\nabla_x w = 0, \{w_{jk}\} \geqslant 0 \text{ at } \bar{x}. \tag{4.21}$$

Furthermore, since in $\overline{V}$, w has a minimum, zero, at $(\overline{x}, \mu)$, we have there $\partial_\lambda w \leqslant 0$.

It follows from (4.14) that $\{w_{jk}\} = 0$ there.

In V we now apply Lemma $\tilde{\mathrm{S}}$ (Lemma 4.3). Here near $(\overline{x}, \mu)$ the function $\rho = \rho(x_2, \ldots, x_n)$, $\rho < 0$ describes ω, and $\sigma = -x_1 - a$. So $a_{ij}\rho_i\sigma_j = 0$ everywhere. We may suppose $(1, 0, \ldots, 0)$ is exterior normal to $\partial\omega$ at $\overline{y}$. By the lemma,

$$(\partial_1 - \partial_2)^2 w > 0 \text{ at } \overline{x} \text{ — contradiction.}$$

Case 5. $\overline{x} = (\mu, \overline{y})$, $\overline{y} \in \partial\omega$. Here $\overline{\overline{x}} = (a, \overline{y})$. Just proceed as in Case 4 using Lemma $\tilde{\mathrm{S}}$ in V at $(\overline{x}, \mu)$.

Thus all cases are impossible and we conclude that $\mu = a$ — so the Theorem is proved. ■

The proof of Theorem 4.1 yields also the following.

THEOREM 4.1$'$. *Let f be as in Theorem 4.1. Let $\underline{u}$, $u \in C^2(\overline{\Omega})$ be solutions of (4.1) satisfying*

$$\underline{u}(x_1, y) \leqslant u(x_1', y) \text{ for } x_1 \leqslant x_1',\ y \in \partial\omega,$$

$$\underline{u}(x_1, y) \leqslant u(a, y),\ \underline{u}(-a, y) \leqslant u(x_1, y) \text{ for } -a < x_1 < a,\ y \in \omega$$

$$\text{and } \forall x_1 \text{ in } (-a, a),\ \exists\, y \in \omega \text{ such that}$$

$$\underline{u}(-a, y) < u(x_1, y)$$

Then $\underline{u} \leqslant u$ in Ω.

A special case of this is

THEOREM 4.1$''$. *Let u and $\underline{u}$ be positive functions on $\Omega = (-a, a)$ belonging to $C^2(\Omega) \cap C(\overline{\Omega})$ and both satisfying*

$$\ddot{u} + f(x, u) = 0 \text{ in } \Omega$$

$$u = 0 \text{ at } \pm a.$$

Here f is continuous in (x, u), Lipschitz continuous in u, and f is symmetric in x, and nondecreasing in x for $-a < x < 0$. Then

(i) u and $\underline{u}$ are symmetric in x and $u_x, \underline{u}_x > 0$ on $-a < x < 0$.

(ii) The functions u, $\underline{u}$ are identical or one is strictly greater than the other in Ω.

Proof: (i) is proved in [4]; (ii) follows from Theorem 4.1$'$.

This result does not hold in higher dimensions, for $n \geqslant 3$ see Lin, Ni [10].

Let us turn now to noncylindrical domains. Consider a bounded domain Ω in R^n with smooth boundary and which is convex in the x_1 direction. Can one extend Theorem 4.1 to this? Under somewhat stronger conditions one can give a very simple proof. Here is such a result.

THEOREM 4.2. *Let Ω be as just described, and assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies (4.1) and (4.2) with f as in Theorem 4.1. Assume that if (x_1', y), $(x_1'', y) \in \partial\Omega$, $x_1' < x_1''$, then*

(4.22) $$\phi(x_1', y) < \phi(x_1'', y)$$

and if in addition $(x, y) \in \Omega$, $x_1' < x_1 < x_1''$, then

(4.23) $$\phi(x_1', y) < u(x, y) < \phi(x_1'', y).$$

Then the conclusion of Theorem 4.1 holds.

Proof: We may suppose that the longest open interval in Ω parallel to e_1 has length 2a and $x_1 = \pm a$ at its right and left endpoints. For $-a < \lambda < a$ let

$$\Sigma(\lambda) = \{x \in \Omega; x + (a - \lambda)e_1 \in \Omega\}.$$

Suppose $\underline{u}$ is a solution of (4.1) in Ω satisfying the same conditions as u. As before it suffices to show (4.6) for $x \in \Sigma(\lambda)$. This then proves the theorem. In the region in (x, λ) space:

$$U = \{(x, \lambda); x \in \Sigma(\lambda), -a < \lambda < a\}$$

w satisfies the parabolic inequality (4.14). On the spatial boundary of U, i.e. the part of the boundary in $\lambda < a$, we have $w \geqslant 0$. In fact for $-a < \lambda < a$ by (4.22),

(4.24) $$w(x, \lambda) > 0 \text{ if } x \in \partial\Sigma(\lambda), x \text{ and } x - (a - \lambda)e_1 \in \partial\Omega.$$

In x-space, on the boundary of each components of $\Sigma(\lambda)$ there is a point x with x and $x - (a - \lambda)e_1 \in \partial\Omega$, otherwise $\partial\Sigma(\lambda)$ would consist entirely of $\partial\Omega$ – impossible. So

(4.25) $$w \geqq 0 \text{ for } x \in \partial\Sigma(\lambda).$$

For $0 < a + \lambda$ small, Proposition 1.1 implies that

$$w(x, \lambda) \geqslant 0.$$

Then by (4.25) and Lemma 4.2, it follows that $w(x, \lambda) > 0$ in U for $0 < a + \lambda$ small.

In $(-a < \lambda < a)$ there is a maximal open interval $(-a < \lambda < \mu)$ for which the inequality (4.17) holds, and we wish to show that $\mu = a$. Suppose $\mu < a$. By continuity $w(x, \mu) \geqslant 0$ for $x \in \Sigma(\mu)$, and again we conclude that

$$w(x, \mu) > 0 \quad \text{for} \quad x \in \Sigma(\mu).$$

Proceed as before: by definition of μ there is a sequence $\lambda^i \searrow \mu$ and points $x^i \in \Sigma(\lambda^i)$ such that (4.18) holds. And we may suppose that in $\overline{\Sigma(\lambda^i)}$, w has its (negative) minimum at x^i. So $\nabla_x w = 0$, $\{w_{jk}\} \geqslant 0$ there. A subsequence $x^i \to \bar{x} \in \overline{\Sigma(\mu)}$, because of (4.22), (4.23) $\tilde{x}^i \to \bar{\bar{x}} = \bar{x} + (a - \mu)e_1$. Clearly $w = 0$, $\nabla_x w = 0$, $\{w_{jk}\} \geqslant 0$ at $(\bar{x}, \mu)$. So $\bar{x} \in \partial\Sigma(\mu)$.

In virtue of (4.23) and (4.24),

$$w > 0 \quad \text{on} \quad \partial\Sigma(\mu), \text{ contradiction.}$$

So in this case we know $\mu = a$ and Remark 4.2 is proved. ■

As before we see that Corollary 4.1 holds. In addition we have the analogue of Cor. 4.2: antisymmetry.

It is natural to ask if one can relax condition (4.22). Here is a result in this direction.

THEOREM 4.3. *Let Ω be as in Theorem 4.2. In place of (4.22), assume for (x_1', y), $(x_1'', y) \in \partial\Omega$, $x_1' < x_1''$*

$$\phi(x_1', y) \leqslant \phi(x_1'', y) \tag{4.22$'$}$$

and if in addition, $(x, y) \in \Omega$, $x_1' < x_1 < x_1''$, then strict inequality holds in (4.22)′, and (4.23) holds. Assume in addition Hypothesis 3): if the segment joining (x_1', y) to (x_1'', y) lies on $\partial\Omega$ and on it ϕ is constant, then the exterior normal ν to $\partial\Omega$ is also constant on it. (This automatically holds if $n = 2$). The conclusion of Theorem 4.1 then holds.

Proof: Proceed as before. Suppose $\mu < a$. Again we obtain a point $\bar{x} \in \partial\Sigma(\mu)$ such that at $(\bar{x}, \mu)$, $w = 0$, $\nabla_x w = 0$, $\{w_{jk}\} \geqslant 0$. Furthermore $\partial_\lambda w \leqslant 0$ at this point so by (4.14) $\{w_{jk}\} = 0$ there. $w(x, \mu) > 0$ for $x \in \Sigma(\mu)$. If $\bar{x}$ belongs to a smooth part of $\partial\Sigma(\mu)$ we have, as before, using Lemma $\widetilde{\mathrm{H}}$, $\partial_\nu w(\bar{x}, \mu) < 0$ – contradiction. So $\bar{x}$ is such that $\bar{x}$ and $\bar{x} + (a - \lambda)e_1 \in \partial\Omega$ and ϕ has the same value at these points. By conditions (4.22)′ and (4.23) it follows that the segment joining these points belongs to $\partial\Omega$.

But then by Hypothesis 3), the exterior normal ν to $\partial\Omega$ is constant on this segment; we may suppose $\nu = e_2$. Because of this, we may apply Lemma 4.4 and get $\partial_\nu w(\bar{x}, \mu) < 0$ – contradiction. Thus $\mu = a$ and the theorem is proved. ■

Suppose the domain Ω is symmetric in x_1 about $x_1 = 0$, and of course, convex in the x_1 direction, and suppose ϕ satisfies the conditions in Theorem 4.3. Then Hypothesis 3) is automatically satisfied. Thus we have

COR. 4.4. ANTISYMMETRY. *Assume the conditions of Theorem 4.3, without, however, Hypothesis 3). Suppose Ω is symmetric in x_1 and that ϕ is odd in x_1. Assume $f(x, u, p)$ is odd in $(x_1, u, p_2, \ldots, p_n)$. Then u is antisymmetric in x_1.*

Appendix. Proofs of Proposition 1.1, of Lemmas 4.1-4.4 et al.

Proposition 1.1 and Lemmas 4.1, 4.2 are well known in case $\beta > 0$ (see e.g. Chapter 3, Sec. 2 of Protter-Weinberger [7]). The proofs for our case $\beta \geqslant 0$ are very similar and will just be sketched. In particular the proof of Proposition 1.1 is essentially the same as the corollary on page 213 of [4].

Proof of Proposition 1.1.: Suppose $b_1 \geqslant -b$, $c \leqslant c_1$ with $b, c_1 \geqslant 0$, and suppose $\epsilon > 0$ so small that

$$c_1 \exp(4b\epsilon/c_0) < c_1 + 2b^2/c_0.$$

Recall that a is the ellipticity constant in (1.8). The function

$$g = e^{\alpha(-a+2\epsilon)} - e^{\alpha x_1}$$

is positive in $\overline{V}$ and satisfies

$$-Lg = (a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} - c(e^{\alpha(-a+2\epsilon)} - e^{\alpha x_1}).$$

Thus choosing $\alpha = 2b/c_0$ we see that

$$\begin{aligned} -e^{-\alpha x_1} Lg &\geqslant c_0\alpha^2 - b\alpha + c_1 - c_1 e^{2\alpha\epsilon} \\ &= \frac{2b^2}{c_0} + c_1 - c_1 e^{2\alpha\epsilon}, \text{ since } \alpha = 2b/c_0, \\ &> 0. \end{aligned}$$

So the function

$$v = w/g$$

satisfies

$$\left(L' + \frac{Lg}{g} - \beta\partial_t\right) v > 0 \text{ in } V$$

where L' is an elliptic operator with no zero order term. Because $Lg/g < 0$ in $\overline{V}$ we see that v can attain a positive maximum only on $J = \overline{\partial V \cap \{t < T\}}$. So $v \leqslant 0$ and hence $w \leqslant 0$.

Note that for all (x^0, T) belonging to $\partial V \setminus J$ there is an open half ball centred at (x^0, T) lying in V.

Proof of Lemma 4.1: By considering a smaller parabolic cap touching $\overline{Q}$ at $(\overline{x}, \overline{T})$ we may suppose that $w < 0$ at every point of $\overline{Q}$ except (x, T). We may suppose $x^0 = 0$ and $\overline{x} = (\overline{x}_1, 0, \ldots, 0), \overline{x}_1 > 0$. In $\overline{Q}$, near $(\overline{x}, T)$, we will construct a C^2 function h, positive in Q, $h = 0$ on the curved part of ∂Q, $h_1 < 0$ at $(\overline{x}, \overline{T})$ and satisfying

$$(L - \beta\partial_t)\, h > 0$$

in the region $\tilde{Q} = Q \cap \{x_1 > \overline{x}_1/2\}$. It follows that for $\epsilon > 0$ sufficiently small,

$$(L - \beta\partial_\tau)(w + \epsilon h) > 0$$

in $\tilde{Q}$, and on the boundary of $\tilde{Q}$ we have $w + \epsilon h \leqslant 0$, except the top, $t = \overline{T}$.

Now the cap Q may be chosen as small as we like, so we find from Proposition 1.1 that $w + \epsilon h \leqslant 0$ in $\tilde{Q}$. Since $w + \epsilon h = 0$ at $(\overline{x}, T)$, necessarily $\partial_\nu(w + \epsilon h) \geqslant 0$ there. But $\partial_\nu h < 0$ there – the result follows.

Now to construct h. Set

$$h = e^{-\alpha(|x|^2 - t)} - e^{-\alpha(\delta - T)}.$$

For α positive, this is positive in P and zero on ∂P, and $\nabla_x h \neq 0$ on ∂P except at the bottom point $(x^0, T - \delta)$. Furthermore

$$(L - \beta\partial_t)\, h = e^{\alpha(t - |x|^2)}\, [4\alpha^2 a_{ij} x_i x_j - 2\alpha \Sigma\, a_{ii} - 2\alpha b_i x_i + c - \beta\alpha] - $$
$$- c e^{-\alpha(\delta - T)} > 0 \text{ in } \tilde{Q} \text{ for } \alpha \text{ large}.$$

The lemma is proved. ∎

Proof of Lemma 4.2.: We may suppose $t^0 = T$. It suffices to show that $w < 0$ on every polygonal path in V_T starting at (x^0, T). We show that $w < 0$ on the first closed straight segment of this path starting at (x^0, T). Continuing this argument one finds that $w < 0$ on the whole path.

We may suppose $x^0 = 0$ and the segment is $S = (x_1, 0, \ldots, 0, T), 0 \leqslant x_1 \leqslant \ell$. Consider a small quarter ball lying in V in the closure of which $w < 0$:

$$\{(x, t); x_1^2 + \ldots + x_n^2 + (T - t)^2 < \epsilon^2,\ t \leqslant T,\ x_1 > 0\}.$$

Now stretch this in the x_1 direction, as a quarter ellipsoid:

$$E_\tau = \{(x, t); t \leqslant T, x_1 > 0, \tau x_1^2 + x_2^2 + \ldots + x_n^2 + (T-t)^2 < \epsilon\}$$

by decreasing τ from 1. As we keep decreasing τ to some τ_0, E_{τ_0} will eventually contain the whole segment S and will still belong to $\tilde{V}$ if ϵ is sufficiently small. We claim that for all τ, $1 \geqslant \tau \geqslant \tau_0$ we have $w < 0$ on $\overline{E}_\tau$ – and so $w < 0$ on S. If not there is a first such τ such that $w < 0$ in E_τ and $w = 0$ at some point z on the curved boundary of E_τ. This point is not the bottom point $(0, T - \epsilon)$ and so we can fit a small parabolic cap in E_τ touching E_τ at the point z. But by Lemma 4.1 $\nabla w \neq 0$ at z. Impossible since z is a point in $\tilde{V}$. ■

Proof of Lemma 4.3. We will follow the proof of Lemma S of [4] – the reader should have that paper in hand. We may assume $x^0 = 0$, $T = 0$.

The conditions of the lemma are invariant under changes of variable $(x, t) \to \to (y(x, t), t)$. If ρ_1, $\sigma_n \neq 0$ at $(0, 0)$ we may introduce new spatial variables $(\rho, x_2 - \chi_2(t), \ldots, x_{n-1}(t) - \chi_{n-1}(t), \sigma)$. The transformed V, near the origin, consists of the region, x_1, x_n, $t < 0$. The curve $\mathscr{C}$ now lies on the t axis. Next, by considering a slightly smaller region $x_1 + \gamma\Sigma_2^{n-1} x_\alpha^2$, $x_n + \gamma\Sigma_2^{n-1} x_\alpha^2$, $t < 0$, here $\gamma > 0$, we have $w < 0$ on its closure, near $(0, 0)$, except possibly on the t axis. Making a second change of variables $(x, t) \to (y, t)$, $y_j = x_j + \gamma\Sigma_2^{n-1} x_\alpha^2$, $j = 1$ and n, $y_\alpha = x_\alpha$, $2 \leqslant \alpha \leqslant n-1$, we obtain: $V = \{x_1, x_n, t < 0\}$ near the origin, and $w < 0$ in $\overline{V}$ except possibly on the negative t – axis. $\mathscr{C}$ is still a segment on the nonpositive t-axis. We're still not through with changes of variables. We wish to have

$$\text{(A.1)} \qquad a_{1\alpha} = a_{\alpha n} = 0 \text{ on } \mathscr{C} \text{ for } 1 < \alpha < n.$$

As in [4] this is achieved by a change of variables: $y = x_1, y_n = x_n$, $y_\alpha = x_\alpha + c_\alpha(t)x_1 + d_\alpha(t)x_n$; see the computation there.

So finally we have near the origin: $V = \{x_1, x_n, t < 0\}$, $\mathscr{C}$ is the nonpositive t – axis, and (A.1) holds.

Now we follow pages 240-242 of [4]. There one took $c = 0$. We will not do that here; we write

$$L = M + c.$$

The inequalities expressed in pages 240-242 of [4] will hold for M. There, two functions

$$\phi = x_1 + k \sum_2^{n-1} x_\beta^2, \quad \psi = x_n + k \sum_2^{n-1} x_\beta^2,$$

were introduced, and a region, in x-space, near the origin,

$$G = \{\phi,\ \psi < 0\}.$$

Now we consider the region in (x, t) space near the origin:

$$\tilde{G} = \{\phi,\ \psi,\ t < 0\} \tag{A.2}$$

On pages 240, 241 of [4] for large k and then large α, the functions

$$z(x) = gh,\ g = e^{-\alpha\phi} - 1\ ,\ h = e^{-\alpha\psi} - 1$$

were constructed. In a small region $G_{\alpha,\delta} = \{-1/\alpha \leqslant \phi,\ \psi < 0\} \cap \{|x| < \delta\}$ the function z was shown to satisfy

$$z > 0,\ \text{and}\ z = 0\ \text{on}\ \phi = 0\ \text{and on}\ \psi = 0 \tag{A.3}$$

$$\partial_\nu z = 0,\ \partial_\nu^2 z > 0\ \text{at}\ 0. \tag{A.4}$$

and

$$\frac{e^{\alpha(\phi+\psi)}}{\alpha^2} Mz \geqslant c_1 \alpha(|\phi| + |\psi|),\ c_1 > 0. \tag{A.5}$$

In verifying (A.4) the properties

$$a_{1n} \geqslant C\left(x_1 + x_n - \sum_2^{n-1} x_\beta^2\right)\ ,\ C > 0 \tag{A.6}$$

$$|a_{1\beta}|,|a_{n\beta}| \leqslant C|x|\ ,\ 1 < \beta < n, \tag{A.7}$$

were used. Because of (4.10) and (A.1), these properties (A.6), (A.7) continue to hold in the region

$$\tilde{G}_{\alpha,\delta} = \{(x, t); x \in G_{\alpha,\delta},\ -\delta < t < 0\}$$

for δ sufficiently small. Thus (A.3) and (A.5) hold in $\tilde{G}_{\alpha,\delta}$.

Consider now in $\tilde{G}_{\alpha,\delta}$ the function

$$\tilde{z} = z + tz^{1/2}.$$

In the region W where it is positive, we have

$$\begin{aligned}(L - \beta\partial_t)\tilde{z} &= Mz + cz + \frac{t}{2} z^{-1/2} Mz - \frac{t}{4} z^{-3/2} a_{ij} z_i z_j \\ &\quad + ctz^{1/2} - \beta z^{1/2} \\ &\geqslant \frac{1}{2} Mz + cz - Cz^{1/2}\end{aligned}$$

since $t < 0$, hence

$$\geqslant \frac{c_1}{2}\,\alpha^3(|\phi| + |\psi|) - c(|z| + |z|^{1/2}) \qquad \text{by} \quad (A.5).$$

Now in $\tilde{G}_{\alpha,\delta}$ we have

$$z = (e^{-\alpha\phi} - 1)(e^{-\alpha\psi} - 1)$$
$$\leqslant e^2\alpha^2\phi\psi$$

since $e^s - 1 \leqslant es$ for $0 < s \leqslant 1$. Hence, with a different constant C,

$$(L - \beta\partial_t)\tilde{z} \geqslant \frac{c_1}{2}\,\alpha^3(|\phi| + |\psi|) - C\alpha^2\phi\psi - C\alpha(\phi\psi)^{1/2}$$
$$> 0 \text{ in } W$$

for α large.

Now we use the function $\tilde{z}$ as a comparison function in the usual way. On ∂W except at the origin we have $w < 0$. Hence for small $\epsilon > 0$ we have

$$w + \epsilon\tilde{z} \leqslant 0$$

on the entire boundary of W lying in $t < T$. For α large, W is as narrow as we like in the x_1 direction. Thus Proposition 1.1 applies and we conclude that

$$w + \epsilon\tilde{z} \leqslant 0 \text{ in } W.$$

But $w + \epsilon\tilde{z} = 0$ at $(0, 0)$, so $\partial_\nu(w - \epsilon\tilde{z}) \geqslant 0$ or $\partial_\nu^2(w + \epsilon\tilde{z}) \leqslant 0$. Since $\tilde{z}$ satisfies (A.4):

$$\partial_\nu\tilde{z} = 0, \; \partial_\nu^2\tilde{z} > 0,$$

the desired result (4.11) follows. ■

Lemma 4.4 is a special case of the following parabolic analogue of Lemma 1.1 of [4] (with $\mu > 0$, so $\theta_0 > \pi/2$, and $p < 2$). We take $(\bar{x}, T) = (0, 0)$.

LEMMA A.1. *Let w and V be as in Lemma 4.4 with $a_{ij} \in C(\bar{V})$. In place of (4.9)′ assume that at (0, 0)*

$$\Sigma\, a_{ij}\rho_i\sigma_j = \mu\sqrt{\Sigma\, a_{ij}\rho_i\rho_j}\,\sqrt{\Sigma\, a_{ij}\sigma_i\sigma_j}$$

for some constant μ; clearly $-1 < \mu < 1$. Set $\theta_0 =$ (arccos)$(-\mu)$. Suppose $p >$ $> \pi/\theta_0$. Let C *be a closed cone in $\{t = 0\}$ with vertex at (0, 0) and such that for some $\epsilon > 0$, $\mathrm{C} \cap \{0 < |x| < \epsilon\}$ lies in V_0. Then there is a positive constant δ and a neighbourhood in* C *of* 0 *in which*

(A.8) $$w + \delta |x|^p \leqslant 0.$$

In particular, if $\mu > 0$ we may take $p > 2$, and it follows that if $w \in C^1(\bar{\bar{V}})$ and $\nabla_x w(0, 0) = 0$ then on any direction s from $(0, 0)$ entering V_0 transversally to the boundary, the second spatial derivatives on w cannot be bounded near the origin. On the other hand if $\mu < 0$, and $w \in C^{[\pi/\theta_0],\alpha}$ for $\alpha > \pi/\theta_0 - [\pi/\theta_0]$, then we may take $p < [\pi/\theta_0] + \alpha$ and conclude that at least one of the derivatives

$$\left(\frac{\partial}{\partial s}\right)^j w, \ j = 1, \ldots, \left[\frac{\pi}{\theta_0}\right]$$

is negative at $(0, 0)$.

Proofs: We follow the proof of Lemma A.1 in [4]. As before we may suppose $w < 0$ except possibly on $\{\rho = \sigma = 0\}$. Proceeding as in [4] we may arrange that near the origin V has the form

$$V = \{(x, t); x_n > 0, \ x_1 > x_n \ \cot \ \theta_0, \ t < 0\}.$$

Choose $k = p\theta_0/\pi > 1$.

Following [4] we also make

$$a_{11} = a_{nn} = 1 \ \text{ at } \ (0, 0).$$

We have $a_{1n} = 0$ there. With $\zeta = x_1 + ix_n$, in [4] we constructed the functions of x:

$$v = Im(\zeta^{\pi/\theta_0})$$

$$z = v^k.$$

Near the origin in V, z satisfies

$$Lz \geqslant c_1 |\zeta|^{p-2}, \ c_1 > 0, \ |z| < C|\zeta|^p.$$

Near the origin, in the region W where it is positive, consider the function

$$\tilde{z} = z + tz^\ell$$

with $1 > \ell \geqslant 0$, $\ell \geqslant 1 - 2/p$. (For the case of Lemma 4.4 where $\mu > 0$ we may take $p < 2$ and $\ell = 0$). We have

$$(L - \beta\partial_t)tz^\ell = t\ell z^{\ell-1} Lz + t\ell(\ell - 1)a_{ij}z_i z_j + ctz^\ell - \beta z^\ell$$

$$\geqslant t\ell z^{\ell-1} L z - C|\zeta|^{p\ell}$$

since $t(\ell - 1) \geqslant 0$. Hence in W,

$$
\begin{aligned}
(L - \beta\partial_t)\tilde{z} &\geqslant (1 + t\ell z^{\ell-1})\, Lz - C|\zeta|^{p\ell} \\
&\geqslant (1-\ell)\, Lz - C|\zeta|^{p\ell} \\
&\geqslant (1-\ell)\, c_1\, |\zeta|^{p-2} - C|\zeta|^{p\ell} \\
&> 0
\end{aligned}
$$

since $p - 2 < p\ell$.

For small positive ϵ we have

$$L(w + \epsilon\tilde{z}) > 0 \quad \text{in} \quad W$$

and $(w + \epsilon\tilde{z}) < 0$ on the boundary of W lying in $t < 0$. By Proposition 1.1 it follows (W is narrow) that

$$w + \epsilon\tilde{z} \leqslant 0 \quad \text{in} \quad W.$$

On $t = 0$ we have therefore

$$
\begin{aligned}
w &\leqslant -\epsilon z \\
&= -\epsilon(Im\, \zeta^{\pi/\theta_0})^k .
\end{aligned}
$$

Thus on any ray $x = -\tau\nu$, $\tau > 0$, on $t = 0$ we have (A.8):

$$w \leqslant -\epsilon c_1\, |x|^p . \qquad \blacksquare$$

We conclude with a result used in the proof of Theorem 1.3.

PROPOSITION A.1. *Assume Hypotheses 1) and 2) of Section 4. Set*

$$J = \overline{\partial V \cap \{t < T\}}.$$

Suppose that on a set A in J, described locally by $\rho = 0$ with $\rho \in C^2$, $\nabla_x \rho \neq 0$ on A, the function w satisfies

(A.9) $$\partial_\nu w \leqslant 0$$

where $\nu(x)$ is a smooth spatial unit vector pointing outside of V, with $\nu_1 \equiv 0$. Assume that $J \setminus A$ is nonempty and that on it, $w \leqslant 0$. If V lies in a sufficiently narrow band: $0 < a + x_1 < \epsilon$, then $w \leqslant 0$ in V.

Proof: Following the proof of Proposition 1.1 given at the beginning of this section we constructed $g(x_1) > 0$ in $\overline{V}$ with $Lg < 0$. Then

$$v = w/g$$

satisfied an inequality of the form (4.7), with new $c < 0$ in $\overline{V}$. Since $\nu_1 = 0$ we see that v continues to satisfy $\partial_\nu v \leqslant 0$ on A and $v \leqslant 0$ on $J \setminus A$. Thus it suffices to prove the proposition in case $c < 0$ in $\overline{V}$, which we henceforth assume. We will continue to refer to the solution as w (rather than v).

Suppose

$$\max_{\overline{V}} w = K > 0.$$

Set $W = w - K$, so max $W = 0$. W satisfies

$$(L - \beta\partial_t) W \geqslant -cK > 0 \text{ in } V.$$

It follows that W can achieve its maximum in $\overline{V}$ only on J. So $W < 0$ in V and W necessarily achieves its maximum at a point on A, since on $J \setminus A$, $W \leqslant -K$. But by Lemma $\widetilde{\mathrm{H}}$ (Lemma 4.1) $\partial_\nu W > 0$ there. Contradiction. ∎

REMARK A.1. It is clear that Proposition A.1 holds in the elliptic case, i.e. no t present. We may not drop the condition $\nu_1 = 0$.

Consider the simple example in $-\delta < x_1 < \delta$, $0 < x_2 < \pi$:

$$w = \cos x_1 \sin x_2$$

satisfies $\Delta w + 2w = 0$, vanishes on $x_2 = 0$ and $x_2 = \pi$ while on the lateral boundaries $x_1 = \pm\delta$ (δ arbitrary small)

$$\partial_\nu w < 0$$

where ν represents the exterior normal.

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